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PRINCIPLES

OF

ARITHMETIC

BY

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PREFACE.

THE object of the present work is to give an account of the principles of Arithmetic, omitting all merely mercantile applications. I have endeavoured, as far as possible, in the explanation of the different methods and results to follow the order of historical development. In doing this I have been guided mainly by Cantor's "Geschichte der Mathematik," the chief work on the history of mathematics. I have also consulted Hankel's "Vorlesungen über die Geschichte der Mathematik," and Nesselmann's "Algebra der Griechen," and have verified many special statements by reference to the original authorities. The conception of the subject as a whole, and many of the details have been taken from the mathematical portions of the works of Auguste Comte and in especial from his last great work the "Synthèse Subjective."

HOMERSHAM COX.

CAMBRIDGE,
August 13th, 1885.

PRINCIPLES OF ARITHMETIC.

INTRODUCTION.

ARITHMETIC is as the word implies the science which treats of the relations between numbers. In order to define this science we must then first of all consider what is the kind of questions to which numbers are applied and what are the fundamental relations which exist between them.

Numbers are used for two purposes, to count and to measure. In the first case numbers are employed to enumerate distinct objects and even thoughts or events; in the second case they express some continuous quantity such as a length or an interval of time in terms of some other quantity of the same kind. The latter use of numbers depends on the former, which has indeed been always the more common and familiar. It is in this former application that we naturally first consider numbers.

A number has no meaning apart from the objects to which it is applied. It is like any other abstract term, such as length or mass, not capable of definition. We must therefore define the property of two groups which are said to contain the same number of objects. It is this, that to every one in the first group there corresponds one and only one in the second group. When it is not possible to compare the two groups together directly, we can compare them indirectly by means of a third group. For it is clear from the definition that two groups will contain the same number of objects when each contains the same number as a third group. Those objects which in the former groups correspond with the same object in the latter group will also correspond with one another. It is therefore possible to adopt a standard set of things once for all and to use this set constantly for the purpose of comparison. The groups we form from the standard set are called specially numbers, and when we are asked the number of objects in any other group it is meant that we are to give that one of these special groups which contains the same number of objects.

We must next see what are the fundamental relations between different numbers. The first of these relations arises from *Addition*. If we join two groups of objects to form a third group the number of objects in the whole group is said to be the sum of the numbers of objects in each of the partial groups, and the formation of this sum is called addition. It is clear that the sum of two numbers is independent of the special objects to which the numbers apply. In other words, if we replace each of the partial groups by another group containing the same number of different objects then the whole number of the new objects will be equal to the whole number of the former. For those objects which corresponded to one another in the partial groups may still be considered to correspond when these groups are joined together. The other fundamental relations between numbers arise from *Subtraction*, *Multiplication*, and *Division*. The definitions of these operations will be given hereafter. The relations arising from them are also independent of special objects, and hence an abstract science of arithmetic dealing with these relations is possible.

Arithmetical theorems may be of two kinds, special or general. They may be either the results of the addition, subtraction, &c. of particular numbers, such as the theorem that 3 and 5 are 8, or they may be general truths relating to numbers, such as the theorem that the product of two numbers is unaltered when the multiplier and multiplicand are interchanged. The resulting division of arithmetic into two parts was made long ago by the Greeks. The former part they called *Logistics* and reserved the name "Arithmetic" for the latter part. We shall begin with the study of the former part, since the more special questions are easier than the more general.

We must first of all shew how to find names for numbers. All that is essential for this purpose is a series of words such as one, two, three, four, or visible signs such as 1, 2, 3, 4 following one another in a definite order. When we wish to give a name to the number of objects in a group we repeat the words one, two, three (or write the signs 1, 2, 3) in their order, saying one and only one for each of the objects in the group. The last word we say will be the required name. This is the ordinary process of counting, and we are really comparing the number of objects with the number of words in the series one, two, three. We take these words in fact as our standard set of things. The names for numbers are not all arbitrary. They are formed on a systematic plan, and the systematic formation of numerical words is called *Numeration.* This will be explained in the first section.

Any combination of numbers by means of Addition, Subtraction, Multiplication or Division might be taken as a name for a new number, and we find as a matter of fact that all these operations have been employed in this way. Thus in Latin two from twenty, in French four twenties, in English half a dozen are names for numbers. It is purely the result of a convention that we use a hundred and five and do not use eight and six as a numerical word. But when once a standard form has been adopted we must reduce all other modes of expression to that form. The method by which this is done is shewn in turn for Addition, Subtraction, Multiplication and Division, in the following sections.

We then come to Arithmetic in the Greek sense of the word, and consider some general theorems relative to whole numbers.

We shall then consider numbers in their application to continuous quantities, and shall examine in succession the two cases of quantities commensurable and incommensurable with the unit of measure. We must explain the notation employed for this class of numbers, and the application of the four fundamental operations.

In the last chapter we shall again return to the properties of whole numbers.

CHAPTER I.

NUMERATION.

We have seen already that to provide names for numbers all that is necessary is a series of words or symbols following one another in a definite order. If however these words or symbols were all arbitrary it would soon become impossible to remember them. The problem then is to express all numbers on a systematic plan by means of a few numbers to which arbitrary names are assigned. But we can only express one number in terms of another by means of the operations of Addition, Subtraction, Multiplication and Division. We must consider then which of these four is the most suitable for the systematic formation of numbers. Subtraction and Division may be put aside at once, since by their means we could only express lower numbers in terms of higher, and this would be open to two objections: 1st, since the smaller numbers are the best known to us we should be expressing the more familiar in terms of the less familiar; 2nd, we could not express higher numbers than the highest to which an arbitrary name had been assigned.

Neither could we express all our numbers by multiplication alone, for then we should want an indefinite series of arbitrary names for those numbers which are not the product of any other numbers, such for example as twenty-three, seventy-one, &c.

These are called prime numbers and will be spoken of later.

It is clear then that we must use addition as a mode of formation of numbers. In fact every number may be considered the sum of smaller numbers and these as the sums of others smaller still, till we arrive at numbers as small as we can imagine.

So that if we give names to the first few numbers—how few does not matter—all others could be expressed by means of addition alone. The names obtained in this way would often be very long, but they can be shortened by the use of multiplication, for the numbers added together may be made for the most part equal, and the addition of equal numbers is multiplication. For example, supposing we wished to express twenty-three in a language which had no arbitrary words beyond six, we could say either 1st, one and two and three and four and five and six and two ; or 2nd, four and four and four and four and four and three ; or 3rd, six and six and six and five ; and the latter two modes could be shortened into five fours and three in the one case and into three sixes and five in the other. Again, the names could be shortened by giving arbitrary signs to some high numbers, just as in weighing a heavy body the weights required will be fewer if some of them are great.

We see then that addition alone is essential for naming numbers, but that multiplication may also be employed.

We will now consider the actual application of these operations first of all in the spoken and then in the written language.

In speaking we use arbitrary words for the first ten numbers. By simply adding these together in pairs we can obtain words up to ten and ten.

The simplest plan is to add the numbers successively to ten, thus, ten and one, ten and two, ten and three... ten and nine, ten and ten. In this way only could we obtain the last two numbers and adding any other numbers such as nine and eight together would be superfluous, since we should merely be giving names to numbers already expressed.

We can obtain ten more numbers by adding one, two,...ten to the last number, thus, ten and ten and one, ten and ten and two, up to ten and ten and ten. This is the most we can obtain by adding together the numbers in threes, for as before it will be useless to add any other numbers together. By the addition of four numbers we can count further ten and ten and ten and one up to ten and ten and ten and ten. But we are naturally led to shorten these names by means of multiplication and to say two tens, two tens and one, two tens and two, three tens, four tens, five tens, &c. Without using more than a single multiplication and a single addition we can count up to ten tens. And we see as in the case of addition that it will be useless to multiply other numbers together, such as nine and eight, for in this way we should obtain nothing new.

Still proceeding on the same principle—that is to say, adding the lower numbers in turn to the highest number already named—we continue ten tens and one, ten tens and two, and so on. But a question will now arise—are we to say ten tens and ten, ten tens and two tens, &c. or instead ten and one times ten, ten and two times ten? The former method has been chosen, and to avoid the inconvenient expressions twice ten tens, three times ten tens, we introduce a new word one hundred, meaning ten tens, and say two hundreds, three hundreds, &c. To count beyond a hundred it will be simply necessary to repeat the former numbers, placing the word one hundred before them, as in one hundred and four tens and seven for example, till we come to two hundreds. We then again repeat the early numbers, placing two hundred before them till we come to three hundreds. We proceed in the same manner with three hundreds, four hundreds, and thus count up to ten hundred. We might even go further, and say two ten hundreds, three ten hundreds, but instead a new word, one thousand, is introduced and we say two thousands, three thousands. We can now express any number up to ten thousands, such as three thousands, five hundreds four tens and seven, without multiplying more than a pair of numbers together. One of the pair always belongs to the numbers one, two, three, four, five, six, seven, eight, nine, and the other to the number ten, hundred, thousand. On this principle we ought to introduce a new word for ten thousand, a myriad for instance, and again another word for ten myriads or one hundred thousand, and so on, as far as we wish to count. This plan was actually adopted by the Hindoos. In European languages, however, no new word occurs till we come to a thousand thousand, and we say twenty-three thousand instead of two myriads three thousands. The word million for a thousand thousand is of late origin. It was introduced according to Hankel in the 14th century from the Italian money market (millone, a ton of gold).

We have thus described the method by which numbers are named. These names are slightly modified in different languages. Thus the words two tens, three tens, &c., are corrupted into twenty, thirty, &c. The general rule of mentioning always the parts of a number in the order of their magnitude, as in three hundred and twenty one, is not followed in English for the numbers between ten and twenty; we say sixteen, not ten and six. The word "and" connecting the different parts of a number is generally dropped, as in two thousand, six hundred, twenty-six; in English, however, it is retained after the hundreds. Also the sign of the plural is dropped after the words hundred, thousand.

These modifications, however, are unimportant, and do not affect the general principle. This is, as we have seen, simply to add at each stage the lower numbers in succession to the highest number of those which have been already formed. Thus when we wish to count beyond a thousand we add to a thousand the numbers previously formed, as in one thousand and one, one thousand and two, &c. The principle is naturally connected with our mode of counting objects laid before us. We then generally after we have reached a certain number, ten for example, place the objects already counted on one side and proceed to count over again one, two, three, &c. These are again put on one side and the process is repeated as often as necessary. This will explain the fact that all languages say one hundred and ten and not eleven tens. The same motives which lead us after counting ten objects to put them on one side and begin again would lead us after counting ten heaps of ten each to form a larger heap of these and putting it on one side to begin counting tens again.

Our system of numeration is then in all that is essential determined by the character of numbers. The only important point that remains undetermined is the number of arbitrary numerical names we originally start with, or to put it in another way, the number of objects we set aside to form a group. There was no necessity for this number to have been ten. Any other number, except of course unity, would have answered the purpose. In fact in dealing out many kinds of hard goods for sale it is usual to count by the dozen and the gross. The objects are dealt out one by one till a dozen is reached. This dozen is then laid on one side and another dozen counted out and so on with three dozen, four dozen up to a dozen dozen or a gross. By repeating these operations two, three, four or more gross could be counted. The total number of objects might then be expressed as three gross, five dozen and six, and this name for the number would take the place of what the reader will easily see is its equivalent, four hundred nine tens and eight.

Why then is ten chosen to be what is called the basis of the system of numeration in almost all languages without exception, or in other words, why do we count by tens? This seems to be connected with the natural habit of reckoning on the fingers, and the explanation is confirmed by many of the names for numbers. Thus the Mexicans used a word meaning hand to denote five and a word meaning man to denote twenty, the total number of both fingers and toes. With them twenty-one was one to the other man, forty was two men. Also in various languages either five or twenty or both are used in subordination to ten as a means of forming numbers: the French quatre-vingt is an example.

We pass now to the written numbers, and have to see how these are formed by addition and multiplication from a few arbitrary signs. We may distinguish three principal systems of numeration.

1. The Roman. In this we have arbitrary signs, M, C, X, I, for one thousand, one hundred, ten, one, and the other numbers are formed by the addition of these symbols: thus, twenty-three is XXIII.

This system was adopted in its simplest form in the Egyptian hieroglyphic writing. The earliest system in use among the Greeks was also fundamentally the same. As however it is not easy to distinguish IIIIIII; IIIIIIII; IIIIIIIII; at a glance some further artifice has to be employed. The Egyptians arranged their units, tens, &c. in special figures, as we do the pips on a pack of cards. The Greeks and Romans had subordinate signs V, L, for five and fifty.

2. The Greek. Arbitrary signs are adopted for one,

two, three...nine, ten, twenty, thirty...ninety, hundred, two hundred, &c. These signs were the letters of the alphabet, α, β, γ...ι, κ, λ... The other numbers are formed by the addition of these symbols: thus twenty-three is $\kappa\gamma$. This system was used by the later Greeks.

3. The Indian or modern. The arbitrary signs 1, 2, 3...9 for the first nine numbers and 0 for the absence of a number are adopted. As in the spoken language, these numbers are multiplied into ten, a hundred, &c. and the results added to make the required number. The distinctive point of the system is that the ten, hundred, &c. are not expressed but indicated by the position of the number which multiplies them. Thus when 2 comes in the second place to the left it means two tens instead of two. Twenty-three is expressed by 23.

We must now consider the origin and connection of these systems.

The first seems to be the simplest conceivable. The most natural mode of representing the number of certain objects that cannot be directly indicated would be to take an equal number of more convenient objects. It is said the spies of the North American Indians would put aside a grain of corn for every man of their enemies, and the heap of corn they brought back would shew roughly the size of the troop. If this method were applied in writing the number of objects would be represented by an equal number of dots or strokes. Such a plan however cannot be called a system. The next step would be to replace a group of strokes—say ten—by a single sign such as X. The change would be naturally suggested by the mode of counting out numbers of which we have already spoken in connexion with the oral language. The Negroes still when they have counted through their fingers place a pebble on the ground and begin again. Obviously the number which was represented by two pebbles and three fingers might be written on paper XXIII. An extension of the process would lead to the creation of the sign C for a hundred, and M for a thousand.

This notation, though clear, has the disadvantage of being very lengthy. Those who frequently made calculations would be led to prefer some more concise even though less simple plan. The most ignorant could write XXX and XXXXX are XXXXXXXX, but a man who had learnt to remember that three and five were eight would prefer to write γ and ϵ are η. In the same way now an expert arithmetician or algebraist would leave out steps that are necessary to the beginner, saving time by the use of his memory. It must have been this reason which induced the Greek to adopt arbitrary signs α, β, γ...ι, κ, λ,... in place of I, II, III,... Δ, ΔΔ, ΔΔΔ,...*.

With this notation we should write $\kappa\gamma$ for twenty-three, and $\omega\lambda\alpha$ for eight hundred and thirty-one. Brevity would be gained, but we should meet with two great disadvantages. The first is the increased number of arbitrary signs. For numbers below a thousand this would present no practical inconvenience. So far only twenty-seven unconnected symbols are required and many more

* The latter signs are known as the Herodian numbers.

can easily be retained in the memory if only they are used frequently. But it would be impossible to create fresh signs indefinitely for one thousand, two thousand ... ten thousand, twenty thousand,... &c. In fact the Greeks never attempted to do so, but used instead special artifices to represent numbers above a thousand.

The second disadvantage is seen only in comparing the Greek system with our own, not in comparing it with the Roman. It is the greater difficulty of performing arithmetical operations. We will anticipate for the moment and consider the Greek method and our method of adding and multiplying two numbers, for example, two hundred and forty-one and five hundred and thirty-six. We should say six and one are seven, three and four are seven, five and two are seven ; at the same time writing down the corresponding figures. But a Greek would say five hundred and two hundred are seven hundred, forty and thirty are seventy. So far the difference is not serious. But in multiplying, while we say five times two are ten, four times three are twelve, &c., a Greek would have to say five hundred times two hundred are a hundred thousand, five hundred times forty are twenty thousand, thirty times forty are one thousand two hundred, &c. That is, he would have constantly to bear in mind the products of ten, a hundred, a thousand, &c. into one another. This is an increased burden to the memory and was felt to be of so much importance when the numbers were high that a special rule was invented for determining the products just mentioned.

Thus we see that from two motives the Greeks would be led to desire an improvement in their system of notation, and there are historical indications that by both these motives they were being led to discover such an improvement. To take the first only; after the Greeks had exhausted the letters of their alphabet in representing numbers up to nine hundred and ninety nine, they used the letters for the units over again with a comma following them to indicate thousands. This comma was omitted when the position of the letter shewed its value, thus β meaning two, $\beta\omega\lambda\alpha$ meant two thousand eight hundred and thirty one. There seems no reason why such a method should not be extended. Why when γ is three should not $\gamma\alpha$ be thirty-one? and when η is eight why should not $\eta\gamma\alpha$ be eight hundred and thirty-one, and $\beta\eta\gamma\alpha$ be two thousand eight hundred and thirty-one? The number of symbols will be much reduced and yet the notation will be as concise as before. Natural as such a step seems it was not taken by the Greeks but by the Hindoos. The numerical notation which we still use seems to have been adopted by the latter people about the year 400 A.D. Since long before this time the Greek and Hindoo civilizations had come in contact, and since the Hindoo astronomy is known to be derived from the Greek, it is possible to suppose that we have here a continuation of the process commenced by the Greeks. The systems of notation we have called Roman, Greek and Hindoo, would thus follow each other in historical as they do in logical succession.

Using then the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, for the first nine numbers we may write 47 for forty-seven, the 4 preceding another number meaning not four but four tens; 247 for two hundred and forty-seven, the 2 preceding two other numbers meaning not two, but two hundred; 2831 for two thousand eight hundred and thirty-one, the 2 preceding three other numbers meaning not two but two thousand. But how is such a number as two thousand and thirty-one to be represented? In order that the 2 may still be in the fourth place to the left, and thus signify two thousand, we must have some symbol to fill up the place where the number of the hundreds would be if there were any hundreds. The symbol employed for this purpose is 0. It simply indicates the absence of units, tens, hundreds, &c. Thus we may write 2031 for two thousand and thirty-one, and in the same way 201 for two hundred and one, 20 for twenty. The simple rule that the value of a symbol is increased ten times by removing it a place further to the left enables us to express every number by means of the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Such is the system of notation invented by the Hindoos, and used without alteration for nearly 15 centuries at least. It is so perfect that we cannot conceive it capable of any further improvement. For in all methods that have been used of expressing numbers, the number is considered to be composed of so many units, tens, hundreds, &c. In the spoken language this constitution is stated in the most obvious way; we say three hundreds and four tens just as we say three apples and four pears. But in the written language besides the methods which we have mentioned as the most historically important various others have been adopted. There must however in all be in the first place some mode of expressing the numbers one, two, three, &c., and in the second place some mode of expressing tens, hundreds, thousands, &c. For the first the most concise method is to use as many arbitrary symbols as there are numbers. The Roman method of writing I, II, III,... is, it is true, even simpler, but it is too long to be applicable to extended calculations. For the second the shortest and simplest way is to write the symbols indicating the number of hundreds, tens, units, &c. one after another in their proper order. As the order already distinguishes hundreds from tens, and tens from units, any further distinction in writing would be a superfluous complication. It follows then that we could not adopt any better method of expressing numbers than the one we now use.

When we speak of the hundreds, tens, and units following one another in their proper order, we imply that there is a definite order of succession for these parts of a number. And in fact for all systems of notation ever adopted the important rule first pointed out by Comte holds: that the larger quantity comes before the smaller in the order of alphabetical writing*. Thus the Romans wrote CCXXXI though they might without

* Cantor (p. 12) ascribes this principle to Hankel, and says, "this discovery does all the more credit to his penetration, as in spite of its great simplicity it had always been overlooked." It was however given by Comte in his *Synthèse Subjective*, p. 125.

confusion have written XXXCCI, and the Greeks wrote σλα not λσα. The chief part of the number is mentioned first, so that the whole is formed by a sort of gradual approximation, just as in stating a distance we mention the miles before the yards, and the yards before the inches. This uniform custom doubtless very much facilitated the invention of our modern notation.

When we say this notation is perfect we refer only to the principle on which it is constructed. The special base *ten* is entirely unconnected with the general principle, and any other number except *one* might have been employed as base. We will mention some numbers that have been proposed or used for that purpose.

The German philosopher Leibnitz suggested the use of the number *two*. In that case the symbols 1, 10, 100, 1000, &c. would each represent a number not ten times but double as great as the preceding, that is to say, they would represent the numbers one, two, four, eight, &c. The series of natural numbers would be written 1, 10, 11, 100, 101, 110, 111, 1000, &c., where we should read 11 not ten and one, but two and one; and 111 not a hundred and ten and one, but four and two and one. Every number would thus be expressed by means of the two symbols 1, 0, and there would be no need of any addition or multiplication table. But these advantages would be far more than counterbalanced by the enormous length of the calculations. For example, one thousand would have to be written with ten figures; thus 1111101000.

Comte proposed to use the number seven as base.

The symbols 1, 10, 100, 1000, &c. would then mean one, seven, forty-nine, three hundred and forty-three, each number being seven times the preceding. The series of numbers would be 1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 20,...30,...40,...50,...60,...100, &c. and we should read 12 seven and two, 234 twice forty-nine, three times seven, and four. The multiplication and addition table would be much shortened, we should only have twenty-five results in each instead of sixty-four as at present to remember, while the length of the calculations would not be much increased.

In connection with the spoken language we have already mentioned the number twelve as a base of the numeral system. We should then read 356 as three gross five dozen and six. Two new symbols for the numbers ten and eleven would be wanted. If we suppose these to be *t* and *e* the series of numbers would be written: 1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1*t*, 1*e*, 20, 21...2*e*, 30,... 40,...50,...60,...70,...80,...90,...*t*0,...*e*0,...100. This base has often been considered the best on account of its divisibility by the numbers two, three, four, six. This however seems a doubtful advantage, and the burden of calculations would be much increased by the increased number of symbols. Short multiplication and addition tables are desirable for two reasons; first because there are fewer results to be remembered, and secondly because in actual practice each result is repeated often and becomes better fixed on the memory.

Lastly, we may mention, on account of its historical interest, the sexagesimal system. There were discovered in 1854, at Senkereh, two tablets in Babylonian writing. Of these one consists of the series of natural numbers as far as sixty with their squares, and the other contains the natural numbers as far as thirty-two with their cubes. Now the square of eight is written (substituting our symbols) 14 and the square of eleven 21. That is to say, 1 in the second place means sixty, and 2 in the second place twice sixty or a hundred and twenty. If this system were uniformly applied no less than sixty distinct symbols would be required. It would of course be impracticable to use so many, and a system identical in its main features with the Roman is used in conjunction with the sexagesimal system in the tablets. Thus in Roman symbols we have the square of nine written IXXI, that is to say 60 and 21 or 81, and the square of ten IXXXX, that is to say 60 and 40 or 100. The tablets date from between 1600 and 2300 B.C., so that we have in them the earliest known instances of position as determining value in arithmetic. There is however no indication of any symbol for 0. It is interesting too, to notice that the mode of measuring angles we still employ is of such early origin.

EXAMPLES.

1. Express by means of multiplication only, the first twenty numbers, using as few symbols as possible.

2. Give instances of popular names for numbers in which division or subtraction has been used.

3. Express two gross, ten dozen and eleven, in terms of hundreds, tens, and units.

4. Express eight hundred and sixty-seven in terms of gross, dozens, and units.

5. In expressing numbers up to a thousand, we require the words one, two, three, four, five, six, seven, eight, nine, and also ten, hundred, thousand; that is to say twelve words in all. Taking in turn each of the other numbers (besides ten) from two to twelve as base, find how many words ought to be used in each case. (Hankel, p. 18.)

6. Express on the Roman system as many numbers as possible with V and I for five and one.

7. With five for basis express on the Greek system the first twenty-four numbers.

8. Express 243 in the binary system (with 1, 0 as symbols).

9. Express the binary symbol 1110101 in decimal notation.

10. Express 921 in the septenary system (with 0, 1, 2, 3, 4, 5, 6 as symbols).

11. Express the septenary symbol 643 in decimal notation.

12. Express the duodecimal symbol 18*e*5 in decimal notation (*e* stands for eleven).

13. In the notation of Leibnitz only two symbols 1, 0 are employed. In the Roman system for numbers up to ninety-nine inclusive only two symbols, X, I, need be employed. Compare the number of figures that would be required in the two cases for writing down all the numbers up to ninety-nine inclusive.

CHAPTER II.

SECTION I.

ADDITION.

ADDITION is the operation of finding the number of objects in two sets of objects taken together when the number of objects in each set is known. The most obvious method of adding is therefore to put the two sets together and to count the whole number. Thus to add seven and five we put seven objects (strokes for example) and five objects together thus,

| | | | | | |, | | | | |

and then count all the strokes. As the seven strokes have been counted already, we may begin counting with the first of the five strokes and say: eight, nine, ten, eleven, twelve. We thus find that seven objects and five objects together make twelve objects, or, as we may say more shortly, seven and five are together equal to twelve.

The obvious rule then for adding two numbers is to count onwards from the first as many numbers as there are units in the second number. This would be very tedious, and often give rise to error if the numbers were at all large. But in that case the mode of naming numbers that has just been explained suggests at once a much better method. We think of a number as made up of units, tens, hundreds, &c. It is natural then, when we have two numbers to add, to add the units together, the tens together, and the hundreds together. For example, if we are to add twenty-three, which we may write XXIII, to forty-five, which we may write XXXXIIIII, we put the units together and the tens together, and thus make XXXXXXIIIIIIII, or in words sixty-eight. Without using written symbols we may say, to add twenty-three to forty-five we first add twenty to forty and get sixty (two tens and four tens are six tens); then we add three to five and get eight; hence the whole sum is sixty-eight.

As another example, we may add two hundred and thirty-four to three hundred and forty-one. The numbers may be written CCXXXIIII and CCCXXXXI, and their sum will therefore be CCCCCXXXXXXXIIIII, that is to say five hundred and seventy-five. In words, two hundred and three hundred are five hundred, thirty and forty are seventy, four and one are five.

In these examples the total number of the units was less than ten, and the total number of tens less than a hundred, but this is not always the case. Suppose, for example, we have to add twenty-seven, XXIIIIIII to thirty-six, XXXIIIIII. The result is XXXXXIIIIIIIIIIIII. But instead of IIIIIIIIIIIII we may write XIII, and we get then XXXXXXIII, or sixty-three. In words, twenty and thirty are fifty; seven and six are thirteen, but fifty and thirteen are sixty-three (five tens and ten and three are six tens and three). Similarly when necessary we replace ten tens by a hundred and ten hundreds by a thousand.

We will see now how we should write a sum in addition in our modern notation.

We take the numbers 2789 and 3425.

Since we have to add the units together, the tens together, &c. we write the numbers so that the units of one fall under the units of the other, the tens of one under the tens of the other, &c. thus:

$$\begin{array}{r} 2789 \\ \underline{3425} \\ 6214 \end{array}$$

Now if we were to add the thousands first we should have to correct the number found whenever there were more than ten hundreds*; and similarly a correction might be necessary if we were to add the hundreds before the tens or the tens before the units. This is not important when the addition is made in words, but would be inconvenient in writing.

We begin then by adding the units and say,

Five and nine are fourteen, ten and four.

We put down 4 in the units' place and say,

Ten and twenty and eighty are one hundred and ten.

* The sum would then be written somewhat as below:

$$\begin{array}{r} 2789 \\ \underline{3425} \\ 5104 \\ \underline{111} \\ 6214 \end{array}$$

We put 1 meaning ten in the tens' place and say,

One hundred and four hundred and seven hundred are one thousand and two hundred.

We put down 2 in the hundreds' place for two hundred and say,

One thousand and three thousand and two thousand are six thousand.

We then put down six in the thousands' place, and the result is complete.

But we see that there is no need to make explicit mention of the tens, hundreds, thousands, &c. Whether we have to add five hundreds and six hundreds, or five tens and six tens, or five units and six units, the result will always be to put down 1 under the 5 and 6 (meaning a hundred, ten, or one, as the case may be) and to add 1 to the numbers in the next place to the left (the 1 meaning a thousand, a hundred, or ten, as the case may be). We may then say in all cases,

5 and 6 are 11, put down 1 and carry 1.

The above example would therefore be worked in practice as follows:

5 and 9 are 14, put down 4 and carry 1.

3 and 8 are 11, put down 1 and carry 1 (the 1 is added mentally to 2 to make 3).

5 and seven are 12, put down 2 and carry 1.

4 and 2 are 6.

Several numbers can in this way be added together at once, as in the following sum:

$$\begin{array}{r} 133679 \\ 205843 \\ 283471 \\ \underline{560345} \\ 1183338 \end{array}$$

It is usual in such a sum as this to say,

> 5 and 1 are 6, 6 and 3 are 9, 9 and 9 are 18, put down 8 and carry 1.

But all that need be said is,

> 5, 6, 9, 18; 5, 12, 16, 23; 5, 9, 17, 23; 2, 5, 10, 13; 7, 15, 18; 6, 8, 10, 11.

When we say 18, we write 8 and mentally add the 1 to the 4 in the next column of figures; when we say 23, we write 3 and mentally add the 2 to the 3 in the next column; and so on.

In working the fifth column the figures 7, 15, 18 mean 7 myriads, 15 myriads, and 18 myriads, or one hundred thousand and eight myriads. But as has already been explained, we may put out of sight altogether the fact that the figures mean so many myriads. Whether they meant myriads, or thousands, or hundreds, we should have to write 8 and add 1 to the next column of figures.

We have now seen how numbers are added. The fundamental artifice simply consists in bringing together the units, the tens, the hundreds, &c. of the numbers to be added. This is such an obvious step that it must have been taken in the very earliest times, by all nations in fact that had arrived at a systematic mode of numeration.

A second artifice of much less importance consists in dropping all explicit mention of the tens, hundreds, &c. This latter improvement only became possible through the invention of the Hindoo numeration. For till then a different symbol had to be written down according as tens, hundreds, &c. were meant.

The principle of addition, like that of numeration, is entirely independent of the number *ten*, which has been chosen as base. For example, if we had to add four gross seven dozen and eight to six gross nine dozen and eleven, we should collect the gross together, the dozens together, and the units together, and say—

> Eight and eleven are nineteen or one dozen and seven.
>
> One dozen and seven dozen and nine dozen are seventeen dozen or one gross and five dozen.
>
> One gross and four gross and six gross are eleven gross.

Hence the sum is eleven gross, five dozen and seven.

Suppose now we had to add the two numbers written below in duodecimal notation (*t* and *e* mean ten and eleven)

$$\begin{array}{r} 85t091 \\ 6e7832 \\ \hline 1355903 \end{array}$$

We should proceed as follows :

> 2 and 1 are 3, write 3.
>
> 3 and 9 are 10, write 0 (10 of course means twelve).
>
> 9 and 0 are 9, „ 9.

7 and *t* are 15, write 5 (seven and ten are seventeen, that is twelve and five).

10 and 5 are 15, „ 5.

7 and 8 are 13, „ 13.

As another example we will add the two numbers 1011010, and 1111101 expressed in the binary notation. In this case 10 means the number two. The sum will stand as follows:

```
 1011010
 1111101
--------
11010111
```

1 and 0 are 1, 0 and 1 are 1, 1 and 0 are 1, 1 and 1 are 10, 1 and 1 and 1 are 11, 1 and 1 and 0 are 10, 1 and 1 and 1 are 11.

Here 1 and 1 and 1 are 11 in the fifth column may be considered an abbreviation for one sixteen and one sixteen and one sixteen are one thirty-two and one sixteen.

The result is easily verified. The first number will be found to be ninety, the second a hundred and twenty-five, and the whole two hundred and fifteen.

The method of addition that we have now explained requires us to know the sums of numbers lower than the basis of numeration. These can always be found by direct counting or, what is better, we can construct a table once for all and learn the results by heart. The table given here shews the sum of any two numbers less than 10.

To find the sums of 6 and 8 for example we look along the 6th row and down the 8th column, and in the square common to both we find 14 the required sum.

To construct the table we divide a square into 9 rows and 9 columns. We then write the numbers 1, 2, 3, &c. along the top and down the left-hand side of the square.

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

Next we write in the first row the numbers onwards from 1 in their order 2, 3, 4, &c. till the row is filled up. In the second row we write the numbers onwards from 2 in their order 3, 4, 5, &c. and so with all the rows. That in this way we shall obtain right results is clear; for adding 3 to 5 is the same as counting 3 numbers on from 5. Now in the 5th row we begin counting numbers on from 5, namely 6, 7, 8, &c., and when we reach the 3rd column we have counted 3 numbers and therefore we shall come to 8 the sum of 5 and 3.

The question is often asked, Of what character is the theorem that 7 and 5 are 12? When we unite 7 objects to 5 objects in order to make 12 objects the union is effected entirely in our own minds. The same objects completely unaltered even in position may be considered 7 and 5 on the one hand or 12 on the other, according as we choose to count them. The statement 7 and 5 are 12 expresses then a mental property and no property of objects. Let us now add the 7 and 5 together gradually. That 7 and 1 are 8 is a definition, and that 8 and 1 are 9 is also a definition. We have then,

7 and 1 .... and 1 are together equal to 9,

where the dotted interval means that the 7 and 1 are to be added first and then another 1 is to be added. Again by definition 2 is 1 and 1. Therefore

7 and 2 are equal to 7 .... and 1 and 1,

where the dotted interval means that 7 is to be added to the 1 and 1. That 7 and 2 are 9 is then simply equivalent to the assertion—using algebraical notation—that

$$(7+1)+1=7+(1+1).$$

This is a case of a general law that is called in algebra the associative law, and is expressed by the equation $(a+b)+c=a+(b+c)$ where a, b, c are any three numbers. Assuming this law, we have $7+2=9$, and it follows next that

$$7+3=7+(2+1)=(7+2)+1=9+1=10.$$

Again, $7+4=7+(3+1)=(7+3)+1=10+1=11$

$$7+5=7+(4+1)=(7+4)+1=11+1=12.$$

Thus the theorem 7 and 5 are 12 simply follows from the definition of the numbers and the continued application of the same law. But this law is involved in the conception of number. It cannot make any difference to the total number of objects formed by 7 objects, 3 objects and 1 object, whether we think of the 3 objects along with the 7 objects or along with the 1 object.

Now supposing the sum of numbers under 10 to be known, what do we assume when we add together two numbers over 10, thirty-two and forty-seven for example? When we say thirty-two and forty-seven are seventy-nine we evidently assume that three tens and two together with four tens and seven are equal to three tens and four tens together with two and seven.

In Algebraical language we assume that

$$(a+b)+(c+d)=(a+c)+(b+d).$$

The principle then is, that the total number of objects is independent of the order in which those objects are mentally arranged, and this includes the associative principle just mentioned. This principle is implied in the very definition of number.

Or we may look at the matter in this way. How do we ascertain the identity of meaning of two numerical words in different languages, twelve and douze for example? Clearly by comparing their places in the two series, one, two, three, &c.—un, deux, trois, &c. Now that douze and twelve are the same is merely a matter of language. But instead of counting one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, we may count one, two, three, four, five, six, seven, seven and one, seven and two, seven and three, seven and four, seven and five. That seven and five are identical with twelve is therefore just as much a matter of language as that douze is identical with twelve. But adding five to seven is simply counting on five numbers from seven, and this we do when we say seven and one, seven and two, &c. up to seven and five.

The proposition that seven and five are twelve is then a necessary consequence of the definitions of the words "seven" "and" "five" "twelve."

SECTION II.

SUBTRACTION.

SUBTRACTION consists in finding the number of objects left when a known number of objects is taken away from another known number of objects. The obvious plan is simply to remove the objects that are to be removed (or others representing them) and then count those that are left. Thus, to take four from thirteen strokes, we draw the thirteen strokes and then separate off four and count the strokes that are left.

| | | | | | | | | | / | | | | |.

We find thus that four strokes taken from thirteen strokes leave nine strokes, and since this is true for all objects we say shortly four from thirteen leaves nine.

We may imagine strokes taken away one by one, thus,

||||||||||||/| |||||||||||/|| ||||||||||/||| |||||||||/||||

and the numbers left will be twelve, eleven, ten, nine. We have then this rule for subtraction: Count backwards from the whole number of objects as many numbers as there are objects to be taken away, the number we come to will be the number of objects left.

If the number to be subtracted is large this rule is impracticable. We then however take away the units of the number to be subtracted from the units of the whole number, the tens from the tens, the hundreds from the hundreds. Thus, to take two hundred and thirty-two from five hundred and sixty-four we say, two hundred from five hundred leave three hundred; thirty from sixty leave thirty; two from four leave two; therefore the remainder is three hundred and thirty-two.

In Roman symbols the operation would stand thus, CCC~~CC~~XXX~~XXX~~II~~II~~.

Sometimes the number of units in the number to be subtracted will be greater than in the whole number. In that case, after we have taken away all the units from the whole number, we shall have to take away some units from one of its tens.

For this purpose we may suppose one of the tens converted into units and joined with the remaining units. Supposing we have to take twenty-five from sixty-two, we think of sixty-two as equivalent to fifty and twelve, and we then say five from twelve leave seven, twenty from fifty leave thirty, therefore the remainder is thirty-seven. In Roman symbols we should replace XXXXXII by XXX~~XX~~IIIIIII~~IIIII~~, and taking away XXIIIII should obtain XXXIIIIIII.

We will now see how subtraction is performed in our modern notation. We will take 2849 from 7561. We write the numbers thus;

$$\begin{array}{r} 7561 \\ \underline{2849} \\ 4712 \end{array}$$

and say, nine from eleven leaves two, forty from fifty leaves ten, eight hundred from fifteen hundred leaves seven hundred, two thousand from six thousand leaves four thousand *. Here we have observed that as we cannot take nine from one we must replace sixty-one by fifty and eleven, and as we cannot take eight hundred from five hundred we must replace seven thousand five hundred by six thousand and fifteen hundred. It is clear, however, without needing to repeat explanations already given, that as in Addition we may omit all explicit men-

* For the same reason as in Addition we begin with the units. If we began with the thousands the sum would stand thus,

$$\begin{array}{l} 7561 \\ 2849 \\ 5722 \\ 4\ \ 1 \end{array}$$

The numbers written above should be rubbed out and replaced by those below them.

tion of the tens, hundreds, thousands, &c. We need then only say,

9 from 11 leaves 2,	4 from 5 leaves 1,
8 from 15 leaves 7,	2 from 6 leaves 4.

In effecting these operations all we require to know is, the remainder left when any of the numbers under ten are subtracted from any of the nine numbers next above them. These results can be obtained from the addition table. To find the difference of 15 and 6 we look along the 6th row till we come to 15; we then look up the column in which 15 is and find 9, the difference required.

There is another slightly different way in which we may look at Subtraction.

Instead of saying, How many objects are left when we take four objects from thirteen objects? we may say, How many objects must we add to four objects in order to make thirteen objects?

To solve the question we should count onwards from four till we come to thirteen; thus, five, six, seven, eight, nine, ten, eleven, twelve, thirteen; and then we should see how many numbers had been counted. We thus find four and nine make thirteen, or, in other words, four from thirteen leaves nine. In the same way, we should ask, What number must be added to 2849 to make 7561? Writing the sum as before we should say 9 and 2 make 11, 5 and 1 make 6, 8 and 7 make 15, 3 and 4 make 7. It will be noticed that a slight difference is introduced in the operation. Before we took 4 from 5 and 2 from 6: now we take 5 from 6 and 3 from 7. The two ways are equally easy, but it is best to adopt uniformly either the one or the other.

The theorems of subtraction are simply the theorems of addition expressed in a different form. The statement that 2849 taken from 7561 leaves 4712 is simply a verbal alteration of the statement that 2849 and 4712 are together equal to 7561. Since then all theorems of addition are necessary deductions from the definitions of the numbers and the definition of addition, it follows that all theorems of subtraction are necessary deductions from the definitions of the numbers and the definition of subtraction.

SECTION III.

MULTIPLICATION.

To multiply one number by another is to add together as many numbers equal to the first as there are units in the second. Multiplication is thus a particular case of addition, and may be effected by the rules for performing the latter operation.

If we wish to multiply 58672 by 4 we write 58672 down four times and add the numbers as below.

$$\begin{array}{r} 58672 \\ 58672 \\ 58672 \\ \underline{58672} \\ 234688 \end{array}$$

The result—called the product of the two numbers—is found to be 234688. But when the multiplier is large the direct addition of so many numbers will be laborious or even impracticable. The operation may be facilitated by arranging the numbers to be added together in groups. Just as in counting objects we arrange them in groups of ten, groups of a hundred, &c., so in adding equal numbers we arrange the numbers as if they were so many objects in groups of ten, groups of a hundred, &c. If we have to multiply 58672 by 24 we write down 58672 twenty-four times in the following manner:

58672	58672	58672
58672	58672	58672
58672	58672	58672
58672	58672	58672
58672	58672	234688
58672	58672	
58672	58672	
58672	58672	
58672	58672	
58672	58672	
586720	586720	

But without performing the operation of addition it is easy to find the sum of ten numbers equal to 58672. In fact it is the principle of our numeration that every symbol is increased in value ten times when moved one place further to the left. We have only then to move every symbol in the number 58672 one place further to the left. This is done by writing a 0 after the whole number, thus, 586720. We have thus multiplied 58672 by ten; since we have increased two to twenty, seventy to seven hundred, six hundred to six thousand, eight thousand to eighty thousand, fifty thousand to five hundred thousand.

We must next add the two numbers equal to 586720 we have obtained; thus

586720
586720
1173440

and after that add the result to the number 234688 we obtained before as the sum of four numbers equal to 58672,

234688
1173440
2408128

We thus find that 2408128 is the sum of twenty-four numbers equal to 58672; or, in other words, 2408128 is the product of 58672 and 24.

In the same way, if we had to multiply 58672 by 324 we should arrange the numbers 58672 in three groups of a hundred each, two groups of ten each, and four numbers over. Now each symbol is multiplied a hundred times by being moved two places further to the left. Hence a hundred times 58672 is 5867200.

We must add together three numbers equal to this.

5867200
5867200
5867200
17601600

We now add this result to the results previously found for twice 586720 and four times 58672.

234688
1173440
17601600
19009728

We thus find that the product of 58672 and 324 is 19009728.

To multiply by any number is therefore reduced to multiplying by a number under ten, and the whole artifice of multiplication consists in this reduction. Now in multiplying 58672 by 4, or in adding together four numbers equal to 58672, all we require to know is, as we saw in Addition, the sum of four numbers each equal in turn to 2, 7, 6, 8, 5. If now we can remember these sums, that is to say if we can remember the products of numbers under 10, we shall avoid the trouble of direct addition. The multiplication of 58672 by 324 would then stand thus,

```
  58672
    324
 234688
117344
176016
19009728
```

and we should say,

4 times 2 are 8, 4 times 7 are 28, 4 times 6 are 24 and 2 are 26, 4 times 8 are 32 and 2 are 34, 4 times 5 are 20 and 3 are 23.

We next multiply by 2, setting down every figure one place further to the left than the corresponding figures in the line above. This renders the operation equivalent to multiplying by 20. It is unusual to write 0 after 117344, for the position of the symbols indicate their value without ambiguity. We need not explain the words used in multiplying by 2.

We next multiply by 3 and write every figure one place further to the left than the corresponding figures in the line above, or two places further to the left than the corresponding figures in the number to be multiplied. This is equivalent to multiplying by 300. We then add the results obtained and find the whole product 19009728.

All we now want is to know the products of any two numbers under 10. These can be found in the multiplication table here given.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The construction will be easily seen. In the 5th row, for example, we write down 5 and constantly add 5 to it, writing down the results in each successive column, till all the columns are filled up. To find 7 times 6 we look down the 7th column and along the 6th row and in the common square we see 42, the required product.

The method does not in the least depend on the number chosen as basis of the numeral system. If we had to multiply 5*t*68 by 4073 in the duodecimal we should write the operation as follows:

```
     5t68
     4073
    15780
   351t8
  1e628
  1e993640
```

and say,

3 times 8 are 20 (twenty-four is two dozen),
3 times 6 is 16 and 2 is 18 (eighteen is a dozen and six),
3 times *t* is 26 and 1 is 27 (thirty is two dozen and six),
3 times 5 is 13 and 2 is 15 (fifteen is a dozen and three).

Or, we may multiply in the binary notation 110 by 101.

The sum will stand thus,

```
    110
    101
    110
  110
  11110
```

We verify the result by finding that the multiplied number is six, the multiplier five, and the product thirty.

The reader will see that the simplicity of multiplication depends on the fact that to multiply by 10 we have only to move all the figures one place further to the left. This property is peculiar to our notation; in the Greek after multiplying by 10 an entirely different set of symbols would have to be written. Hence it is in multiplication, as has been already pointed out, that the special advantages of our system over all preceding ones first become apparent.

SECTION IV.

DIVISION.

To divide one number by another is to find the number which when multiplied by the second number becomes equal to the first, or as nearly as possible equal while not exceeding the first.

In more concrete language we may say, If objects are divided into classes, each class containing an equal number of objects, division consists in finding the number of objects in each class when the whole number and the number of classes are given. The whole number of objects is called the dividend, the number of classes the divisor, and the number of objects in each class the quotient. It will generally be impossible to distribute all the objects equally among the classes. After as many objects as possible have been allotted to each class some objects will be left over. The number of these objects, which will of course be always less than the number of classes, is called the remainder.

As in the former operations we begin by considering the most natural and obvious way of performing division.

The method constantly used in dividing cards &c. among several persons is to deal out one card to each person in turn, then another, and so on till the number of cards left is less than the number of persons. Suppose now we wish to divide 12685 loaves of bread among 3947 persons.

We give a loaf of bread to each person, and find by subtraction that there are

$$\begin{array}{r} 12685 \\ \underline{3947} \\ 8738 \end{array}$$

8738 loaves left. We now deal out another loaf to each person and then another, and find that there are successively 4791 and 844 loaves left—

$$\begin{array}{r} 8738 \\ \underline{3947} \\ 4791 \end{array} \qquad \begin{array}{r} 4791 \\ \underline{3947} \\ 844 \end{array}$$

We are thus able to give 3 loaves to each person, and we should therefore say that the division of 12685 by 3947 gives 3 for quotient and 844 for remainder. The obvious rule then for dividing one number by another is to subtract the latter number as many times as possible from the former. The number of subtractions will then be the quotient, and the number left after the last subtraction will be the remainder.

When the quotient is large the direct application of this method will be troublesome. But it may be easily shortened. Suppose we have to divide 83685 loaves among 3947 people. From what has been already said we know that ten times 3947 is 39470, and this is less than 83685. We may then at once deal out 10 loaves to each person and shall thus give away altogether 39470 loaves,

83685	4745
39470 − 10	3947 − 1
44215	798
39470 − 10	
4745	

Performing the subtraction we find that there are 44215 loaves left. We may then deal out 10 loaves more to each person and there will be 4745 loaves left. We next give one loaf to each person and there will be 798 loaves left. We have now altogether dealt out 21 loaves to each person. Hence, when 83685 is divided by 3947, the quotient is 21 and the remainder is 798.

As another example we will divide the same number 83685 by 375. Since 83685 is greater than 37500 we can deal out at once 100 objects to each person. The sum will stand thus,

83685	8685	1185
37500 − 100	3750 − 10	375 − 1
46185	4935	810
37500 − 100	3750 − 10	375 − 1
8685	1185	435
		375 − 1
		60

The quotient is found to be 223 and the remainder 60.

We have seen now how Division can be performed by means of Subtraction, just as Multiplication can be performed by means of Addition.

have stood for twenty, if there had been three it would have stood for two thousand. But about this we need not concern ourselves. Since we write a figure of the quotient for every figure of the dividend after 836, at the end of the operation the 2 will be sure to stand for the right quantity whether it be twenty, two hundred, or two thousand.

EXAMPLES.

1. Count the even and odd numbers forwards and backwards up to 100. Count in the same way any numbers with a constant interval, such as 2, 9, 16, 23... up to 100 and back. These exercises should be practised till they can be performed as fast as the words can be spoken.

2. Taking any series of numbers, such as 82505597 62668536257861064778O389, form, beginning with any number, the sums obtained by adding the succeeding numbers. Thus, beginning at 8 we shall have the sums 8, 10, 15, 20, 25, 34, 41, &c. This is to be done rapidly and in words.

3. Add, beginning with the figures in the highest places, the numbers 4827147, 5186991, 2293656, 2143812.

4. Subtract, beginning with the figures in the highest places, 6697775 from 7926321.

5. Without using the multiplication table multiply 229365 by 2063.

6. Without using the multiplication table divide 389748 by 263.

7. Add in binary notation 11110101, 10110011, 11010110, 10001011.

8. Subtract in the same notation 10110011 from 11110101.

9. Multiply 100011 by 1110.

10. Verify these examples by reducing to decimal notation.

CHAPTER III.

PROPERTIES OF NUMBERS.

SECTION I.

THEOREMS RELATING TO ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION.

WE have said already in the Introduction that Arithmetic contained two parts, a part called by the Greeks Logistics relating to the expression of numbers in a standard conventional form, and another part for which the Greeks reserved the name Arithmetic relating to the establishment of general theorems concerning numbers. We have considered the former part as far as relates to operations on whole numbers. We must now turn to the latter part before commencing the study of fractions. We shall in this section first of all state explicitly and explain the fundamental theorems concerning Addition and Multiplication. These theorems have indeed as we shall shew been already implicitly assumed in the performance of those operations.

The fundamental theorems concerning Addition are

I. The Commutative Theorem. By this is meant that when we have two numbers to add together it is indifferent whether we add the first on to the second or the second on to the first. For instance, if we have 6 and 3 to add we may either count 3 numbers after 6, thus 7, 8, 9, or we may count 6 numbers after 3, thus 4, 5, 6, 7, 8, 9. The result will be the same in both cases, namely 9.

If the first process be meant by the words 6 and 3, and the second process be meant by the words 3 and 6, the theorem may be stated in the shortest manner, thus

6 and 3 are equal to 3 and 6.

That this must be so is of course obvious. If we join 6 objects to 3 objects, whether we think of the 6 objects first or the 3 objects first, the whole number will be the same.

II. The Associative Theorem. *When the number to be added is increased the sum is increased by an equal amount.*

Thus suppose we have to add 3 to 6, if we increase the 3 by 2 and make it 5 we shall increase the sum 9 by 2 and make it 11. If we indicate the numbers to be immediately added by writing them close together and separating by a line from the number to be afterwards added we may write the theorem thus:

6—3, 2 and 6, 3—2 are equal.

Combining this with the commutative theorem the same number may be expressed as the sum of 6—2, 3, 6, 2—3 or 8 and 3.

Another way of stating the associative principle (a way which however assumes the commutative principle) is this:

If two numbers are added together, when one of them is increased and the other diminished by the same amount the sum is unaltered.

The two theorems combined give the following which includes them both.

If several numbers be added together their sum is not affected by the order in which they are added.

We now come to the theorems relating to multiplication. The first of them connects multiplication with addition and is

III. The Distributive Theorem. 1. *Any multiple of the sum of two numbers is the sum of the same multiples of each of the numbers.* 2. *The product formed by multiplying one number by the sum of two other numbers is the sum of the products that would be formed if the first number were multiplied separately by each of the two latter numbers.*

According to (1) 3 times 9 is equal to the sum of 3 times 4 and 3 times 5.

In fact 3 times 9 is the sum of the numbers 4, 5—4, 5—4, 5. But by the theorems relating to addition we may rearrange the parts of a sum, and write the numbers thus: 4, 4, 4—5, 5, 5; and in this form the sum is equal to 3 times 4 together with 3 times 5.

According to (2), 9 times 3 is equal to the sum of 4 times 3 and 5 times 3.

This is even more obvious. In fact nine threes are four threes and five threes, just as nine apples are equal to four apples and five apples, nine books to four books and five books, &c.

These three theorems are implied in the operation of Addition. We may convert twenty-five and thirty-four into fifty-nine by the following steps:

twenty-five—thirty-four
twenty-five, thirty—four by II.
twenty, thirty, five—four by I. and II.
fifty-five—four by III.
fifty—nine by II.

The proof that 7 and 5 are 12 we saw depended on the continued application of II.

The next theorem relating to Multiplication is

IV. The Commutative Theorem. The product of two numbers is unaltered by interchanging the multiplying and the multiplied number.

That is to say 7 times 5 is the same as 5 times 7.

This theorem is not quite so obvious as the preceding one but it is easily proved. Suppose we have 7 boys with 5 apples each, then the total number of apples will be 7 times 5. Now we take away all the apples and give back each boy his 5 in the following manner; we first of all give each boy an apple and thus give away altogether 7 apples; we then give away another 7 apples giving each boy one apple and so on. By the time each boy has his

5 apples we shall have given away 7 apples 5 times. Hence 5 times 7 is the same as 7 times 5.

This may also be shewn by arranging dots as follows:

. . . .
. . . .
. . . .
. . . .
. . . .

Here the whole number of dots consists of 7 columns containing 5 each, and also of 5 columns containing 7 each.

Owing to this law the two forms of the distributive law become equivalent.

V. The Associative Theorem. *When two numbers are multiplied together, if the multiplier be increased any number of times the product is increased the same number of times.*

For instance 6 times 7 is 42. Now increase the 6 5 times and we have 30 times 7, and this will be equal to 5 times 42.

To prove this arrange the 7s as here shewn, and imagine them added.

7, 7, 7, 7, 7, 7
7, 7, 7, 7, 7, 7
7, 7, 7, 7, 7, 7
7, 7, 7, 7, 7, 7
7, 7, 7, 7, 7, 7

The first row is 6 times 7 or 42. Therefore the whole number is 5 times this or 5 times 42. But the whole number of 7s is 30. Hence 30 times 7 is equal to 5 times 42. Or we may say at once five times six sevens are thirty sevens just as five times six books are thirty books, &c. The associative theorem and the second part of the distributive theorem unlike the commutative theorem and the first part of the distributive theorem do not involve a rearrangement of the numbers to be added, or in other words do not depend on the associative and commutative theorems for addition.

The associative theorem for multiplication may be stated in the following way (a way which however involves the commutative theorem).

If the multiplying numbers be increased and the multiplied number diminished an equal number of times the product is unaltered.

By combining the two theorems we arrive at the following result which includes them both.

When several numbers are multiplied together the product is unaltered by the order in which they are multiplied.

These theorems are implied in the multiplication of two numbers. Thus in multiplying 46 by 32, we first of all multiply by 2 and then by 30. This involves III. (2). When we multiply 46 by 2, we multiply the 6 and afterwards the 40 according to III. (1). Lastly in multiplying 32 by 40, we make 40 times 32 equal to 4 times 320 according to IV. and V.

We shall often have occasion hereafter to refer to these theorems I. II. III. IV. V. We shall see that they form the basis of all arithmetical and algebraical reasoning.

With reference to Subtraction and Division we may point out that there are two ways in which these questions may be put. When we wish to subtract 3 from 9 we may either ask (1), What number must 3 be added to in order to make 9? or (2) What number must be added to 3 in order to make 9? That the answer to the two questions will be the same is clear from the commutative law of addition (I.), but each question indicates a distinct method of performing the operation. The first would lead us if we had 9 objects to remove the 9th, the 8th and the 7th, and leave 6 objects. For in reversing this process we should put back the 7th, the 8th and the 9th; that is we should add 3 objects to 6 objects. The second would lead us to take 3 objects and then add a 4th, a 5th, a 6th, a 7th, an 8th, a 9th. In this way we shall be adding 6 objects to 3 objects. The distinction between the two methods has already been pointed out in *Subtraction.*

There are also two modes in which the question of division may be put. We may either say (1) What number must be multiplied by 5 in order to make 20? or (2) What number must 5 be multiplied by in order to make 20? The distinction is better seen if the questions be put in a more concrete form. We may say (1) If twenty books be divided into 5 equal portions how many books will there be in each portion? or (2) If there be 5 books in each portion how many portions must be taken in order to make up 20 books? In the first question we divide a concrete number 20 books by an abstract number 5, and obtain as answer a concrete number 4 books. In the second question we divide a concrete number 20 books by a concrete number 5 books, and obtain as answer an abstract number 4. The first is the most natural way of looking at division. The second is the way we look at division when we come to perform the arithmetical operation. The difference is important in the case of fractions.

From the fundamental properties of Addition and Multiplication many properties of Subtraction and Division may be derived. Some of these will be given in the examples.

As an example of the properties of multiplication we may prove the following simple rule for the multiplication of numbers between 10 and 20:

Add the numbers standing in the units' place and multiply the sum by 10. To this result add 100 and to the whole add the product of the numbers in the units' place. The sum will be the product required.

Thus to multiply 17 and 18 add 7 and 8 and multiply by 10. The result is 150. Add 100 and we have 250. Add 56 the product of 7 and 8 and we have 306, the required product of 17 and 18.

In fact 17 times 18 is equal to 10 times 18 together with 7 times 18.

Now 10 times 18 is equal to 100 and 80.

And 7 times 18 is equal to 70 and 56.

Therefore the whole product is equal to the sum of the four numbers

100, 80, 70, 56,

or to the sum of 100, 150, 56,

and these are the numbers the rule requires us to add.

SECTION 2.

Greatest Common Measure.

The Greatest Common Measure of two numbers is the greatest number that will divide them both without remainder.

Thus the greatest common measure of 9 and 15 is 3. For 3 divides 9 exactly giving 3 for quotient, and also divides 15 exactly giving 5 for quotient. But no higher number than 3 divides both 9 and 15. Similarly 13 is the greatest common measure of 39 and 104 since 13 is the greatest number that divides both 39 and 104 exactly.

Before we shew how to find the greatest common measure of two numbers we must first prove two theorems relating to division.

1. If one number is divisible by another then any multiple of the first will also be divisible by the second.

This is really included in theorem (V) of the preceding section. For example, 39 is a multiple of 13 and contains 13 three times. It follows that double, treble, quadruple 39 &c. will contain 13 twice three, three times three, four times three times &c. respectively. We may write 39 thus (13, 13, 13). Then any multiple of 39, for example five times 39, may be written

(13, 13, 13) (13, 13, 13) (13, 13, 13) (13, 13, 13) (13, 13, 13), and is thus seen to consist of a sum of 13s and to be therefore divisible by 13. So any other number divisible by 13 could be represented by a heap of 13s, and any multiple of this number could be represented by several equal heaps. The multiples would therefore also consist of 13s.

2. If two numbers are both divisible by another number, then their sum or difference is also divisible by that number.

This is really included in the distributive theorem (III.) of multiplication. 65 is a multiple of 13 being 5 times 13, and 39 is a multiple of 13 being 3 times 13. The sum 104 will be 8 times 13 and the difference 26 will be twice 13. So that both sum and difference are multiples of 13. As before, any other two multiples of 13 might be represented by two heaps of 13s and the sum and difference of those two multiples would also be represented by heaps of 13s.

As it is very important these theorems should be understood we will illustrate them in another way. We will suppose our numbers to be represented by so many pence. Then any number divisible by 12 will be represented by a certain number of shillings. Now the multiple of any number of shillings is also a certain number of shillings, and the sum and difference of two numbers of shillings will likewise consist only of shillings. If we reduce these shillings to pence we shall be led to say— The multiple of any number divisible by 12 is also divisible by 12, and the sum and difference of two numbers divisible by 12 are themselves divisible by 12.

We will now endeavour to find the greatest common measure of 63 and 140. We first of all find the quotient when 140 is divided by 63. This is 2. Now 63 multiplied by 2 is 126 and any number which divides 63 exactly will divide 126. Take 126 from 140 the difference is 14, and any number which divides 126 and 140 will divide 14. Hence the numbers which divide 140 and 63 divide also 14. And conversely the numbers which divide 63 and 14 divide also 140. For any number dividing 63 must divide 126, and any number dividing 126 and 14 must divide their sum 140.

We must now then find the highest number which divides both 63 and 14. We repeat the former operation. The quotient is 4; 4 times 14 is 56; and the difference of 63 and 56 is 7. By the same reasoning as before we shew that any number which divides both 63 and 14 divides 7, and any which divides both 7 and 14 divides 63. Now 7 divides both 7 and 14, and clearly no greater number than 7 can divide 7 itself. Hence 7 is the greatest number which divides both 63 and 140, or, in other words, 7 is the greatest common measure of 63 and 140.

We may write the operation thus

```
63) 140 (2
    126
    ---
     14) 63 (4
         56
         --
          7) 14 (2
             14
             --
```

We divide 140 by 63 and the remainder is 14; we next divide 63 by 14 and the remainder is 7. 7 divides 14 without remainder and is therefore the greatest common measure.

As another example let us find the G.C.M.* of 5872 and 7000.

The work will stand thus:

```
5872) 7000 (1
      5872
      1128) 5872 (5
            5640
             232) 1128 (4
                   928
                   200) 232 (1
                        200
                         32) 200 (6
                             192
                               8) 32 (4
                                  32
```

The reasoning implied is briefly

The G.C.M. of 7000 and 5872 is the G.C.M. of 5872 and 1128.

The G.C.M. of 5872 and 1128 is the G.C.M. of 1128 and 232.

The G.C.M. of 1128 and 232 is the G.C.M. of 232 and 200.

The G.C.M. of 232 and 200 is the G.C.M. of 200 and 32.

The G.C.M. of 200 and 32 is the G.C.M. of 32 and 8.

The G.C.M. of 32 and 8 is 8, the number required.

* G.C.M. is used as an abbreviation for greatest common measure.

If we wish to find the highest number which divides three given numbers we may find the highest number which divides any two of them and then find the highest number which divides both the number just found and the third given number. Thus, if we want to find the G.C.M. of 150, 45 and 63, we find first the G.C.M. of 45 and 150.

```
45) 150 (3
    135
    ---
     15) 45 (3
         45
         --
```

It is 15. We next find the G.C.M. of 15 and 63.

```
15) 63 (4
    60
    --
     3) 15 (5
        15
        --
```

It is 3. 3 is then the required G.C.M. of the numbers 150, 45, 63. In fact any number which divides 150 and 45 must divide 15. Therefore any number which divides 150, 45 and 63 must divide 15 and 63. But 3 is the highest number which divides 15 and 63. Hence 3 is the highest number which divides 150, 45 and 63.

The rules for finding the G.C.M. of two numbers and of three or more numbers are given by Euclid in the 2nd proposition of the 7th book of his "Elements." The method is placed here in the theoretical part of Arithmetic because unlike the four fundamental operations it has no relation to any mode of numeration.

SECTION 3.

Prime Numbers and Composite Numbers.

A *Prime Number* is a number which can only be divided by itself and by unity.

Thus the numbers 1, 2, 3, 5, 7, 11, 13, 17, &c. are prime numbers.

A number divisible by other numbers besides itself and unity is called a *composite number.*

The numbers 4, 6, 8, 9, 10, 12, 14, 15, 16, &c. are composite numbers.

If one number divides another number exactly, the first number is said to be a *factor* of the second number.

Thus 2 is a factor of 4, 2 and 3 are factors of 6, 3 and 5 are factors of 15.

Any composite number can be expressed as a product of prime numbers, and these are called its prime factors.

For since a composite number is divisible by some other number, the other number and the quotient form two factors whose product is equal to the original number.

These two factors in their turn if not prime can be resolved into other factors, and we may proceed in this way till we have only prime factors.

Thus 1062 is divisible by 2. On dividing we find 1062 to be the product of 2 and 531.

In its turn 531 is found to be the product of 9 and 59. Lastly, 9 is the product of 3 and 3.

Therefore 1062 is the product of the four prime numbers 2, 3, 3, 59.

How do we know that 59 is a prime number? We try the numbers in succession to see if any will divide it exactly. 2 will not, 3 will not. There is no need to try 4, for clearly a number which was divisible by 4 would have been divisible by 2. 5 will not divide 59. There is no need to try 6, for a number which was divisible by 6 would have been divisible by both 2 and 3. 7 will not divide 59. For the same reasons as before there is no need to try 8, 9 or 10. We can easily see that 11 will not divide 59, but we may shew at once that there is no need to try 11 or any higher number. For since 11 when multiplied by itself gives 121, a number greater than 59, it is clear that 59 when divided by 11 or any higher number must give for quotient a number less than 11. This number must either be prime or else composed of prime factors which of course will be less than 11. So that if 59 were divisible exactly by 11 or any higher number, it would be divisible by some prime number lower than 11. But we have shewn that this is not the case. It follows then that 59 is divisible by no other number than itself and unity, and is therefore a prime number.

As a further illustration of the method of trying whether a given number is prime or not we will take 173. We find that 17 multiplied by itself gives 289, a number greater than 173. We need then only try the prime numbers less than 17. These are 2, 3, 5, 7, 11, 13. On trial none of these numbers is found to divide 173.

173 is therefore a prime number.

Now let us find the prime factors of 1105. We try in succession the prime numbers 2, 3, 5, &c. to see which will divide 1105. We find that 5 will and the quotient is 221. We find again that 13 is the first prime number that will divide 221 and 17 is the quotient. 1105 is therefore the product of 5, 13, and 17.

There is a method of finding all the prime numbers below a given number known as the "Sieve of Eratosthenes*," from the name of its inventor.

Suppose we wish to find all the prime numbers up to 50. We write down all the odd numbers less than 50, beginning at 3; thus,

3, 5, 7, ~~9~~, 11, 13, ~~15~~, 17, 19, ~~21~~, 23, ~~25~~, ~~27~~, 29, 31,
~~33~~, ~~35~~, 37, ~~39~~, 41, 43, ~~45~~, 47, ~~49~~.

We then count 1, 2, 3, beginning with the 5 and strike out every 3rd number. In this way we eliminate all the numbers divisible by 3. Next we count on from 5 beginning at 7 and strike out every 5th number, not omitting those that have been struck through already. Lastly, we begin counting at 9, and strike out every 7th number. The numbers now left are not divisible by 2, 3, 5 or 7 and they are less than 50. Hence they must be prime numbers. Joining the numbers 1, 2 on to them we have for the series of prime numbers up to 50,

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

* Eratosthenes was born 275 B.C. and died 194 B.C. He passed the greater part of his life at Alexandria and is chiefly distinguished for his attempt to measure the diameter of the earth.

We need really only begin at 25 in striking out numbers divisible by 5, and at 49 in striking out numbers divisible by 7. This modification would save some trouble when the method is applied to finding a long series of prime numbers. Without going further than striking out 7th numbers we can find all the prime numbers up to 121 and without going further than striking out 11th numbers we can find all the prime numbers up to 169.

The reader will easily see the propriety of the name "Sieve."

Two numbers are said to be prime to one another when they have no common factor except unity.

Thus 24 and 35 are prime to one another. If we perform the operation for finding the G.C.M. on such numbers the last divisor must be 1, since only 1 divides both numbers. With 24 and 35 the operation stands thus:

```
24) 35 (1
    24
    11) 24 (2
        22
         2) 11 (5
            10
             1) 2 (2
                2
```

We may prove from this an important theorem.

If a number is prime to each of two other numbers it is also prime to their product.

For example, 24 is prime to 17 and 35, and it is to be proved that 24 is prime also to their product 35 times 17.

Any number which divides 24 must divide 24 times 17, and if the number divides 35 times 17 also, it will divide the G.C.M. of 24 times 17 and 35 times 17. Now if we multiply by 17 the successive remainders and divisors that occur in the operation of finding the G.C.M. of 24 and 35, we shall have performed the operation of finding the G.C.M. of 17 times 24 and 17 times 35. Since then the last divisor in the former operation was 1, the last divisor in this operation is 17, and 17 is therefore the G.C.M. of 17 times 24 and 17 times 35.

The work will stand thus:

```
408) 595 (1
     408
     ---
     187) 408 (2
          374
          ---
           34) 187 (5
               170
               ---
                17) 34 (2
                    34
                    --
```

Here any number which divides both 24 and 17 times 35 will also divide 17.

But there are no numbers which divide both 24 and 17; therefore there are no numbers which divide both 24 and 17 times 35, or in other words 24 is prime to the product of 17 and 35.

The proof will perhaps be clearer if we repeat the reasoning given in the explanation of the rule for finding the G.C.M. of two numbers.

It is required to find the common factors if any of 24 and 17 times 35.

Any factor of 24 will be also a factor of 17 times 24.

Any factor of 17 times 35 and 17 times 24 will be a factor of their difference 17 times 11.

Any factor of 17 times 11 will be a factor 17 times 22.

Any factor of 17 times 22 and 17 times 24 will be a factor of 17 times 2.

Any factor of 17 times 2 will be a factor 17 times 10.

Any factor of 17 times 10 and 17 times 11 will be a factor of their difference 17.

Therefore the common factors of 24 and 17 times 35 are also common factors of 24 and 17.

But these last numbers have no common factors. Therefore 24 and 17 times 35 have no common factor and are prime to one another.

Exactly the same reasoning shews that when we wish to find the G.C.M. of any two numbers we can divide either of these numbers by any factor prime to the other. For instance, suppose we wish to find the G.C.M. of 24 and 60 times 35. By simply changing the word 17 into 60 in the preceding passage we can prove that the G.C.M. of 24 and 60 times 35 is the same of the G.C.M. of 24 and 60. Here 35 a factor of the second number is prime to 24 the first number, and in consequence we can divide the second number by 35.

We can often by this means materially shorten the process of finding the G.C.M. of two numbers. We may take the numbers before given 5872 and 7000 as an illustration.

Since 7 is prime to 5872 the G.C.M. of 5872 and 7000 is the same as the G.C.M. of 5872 and 1000.

Since 5 is prime to 5872 the G.C.M. of 5872 and 1000 is the same as the G.C.M. of 5872 and 200.

Dividing twice again by 5 the question is reduced to finding the G.C.M. of 5872 and 8.

Now 5872 is divisible by 8 and 8 can be divisible by no greater number than itself.

Hence 8 is the G.C.M. required.

Again, let us find the G.C.M. of 504 and 70.

504 is divisible by 9 and 9 is prime to 70, therefore the G.C.M. of 504 and 70 is the same as the G.C.M. of 56 and 70. 70 is divisible by 5 and 5 is prime to 56. Therefore we have to find the G.C.M. of 56 and 14. Now 14 divides 56.

Hence 14 is the G.C.M. required.

We may prove also in the same way that if two numbers be both multiplied by the same number their G.C.M. will also be multiplied by that number.

For example, the G.C.M. of 49 and 63 is 7; the G.C.M. of 5 times 49 and 5 times 63 will be 5 times 7. In fact to find the G.C.M. of the last pair of numbers we have only to multiply by 5 the divisors, dividends and remainders obtained in the operation for finding the G.C.M. of the first. The work in the two cases will stand thus :

49) 63 (1	245) 315 (1
49	245
14) 49 (3	70) 245 (3
42	210
7) 14 (2	35) 70 (2
14	70

We have now another means of simplifying the operation of finding the G.C.M. of two numbers. We divide the numbers by any factor common to both and afterwards multiply by that factor the G.C.M. of the two quotients. The process of division by common factors may be repeated and may often easily be continued till the final quotients are seen to have no further common factor. In that case the G.C.M. of the original numbers will be the product of all the factors by which we have divided.

Take the numbers 5872 and 7000 once more.

They are both divisible by 4, so we must find the G.C.M. of 1468 and 1750.

These numbers in their turn are divisible by 2 and the quotients are 734 and 875.

Now we can easily shew that 734 and 875 have no common factors. For 734 is prime to 5 and therefore to 25 the product of 5 and 5. Being prime to 5 and 25, 734 is prime to their product 125. Lastly, 734 is prime to 7 and therefore to 875 the product of 7 and 125. Since then the final quotients 734 and 875 have no common factor the G.C.M. of the original numbers 5872 and 7000 is 8 the product of 2 and 4.

Both these modes of simplification may be combined. We may break up each of the numbers into prime factors and then select those factors which are common to both, and their product will be the G.C.M. of the two numbers. For instance 420 is the product of the prime numbers 2, 2, 3, 5, 7 and 1617 is the product of the prime numbers 3, 7, 7, 11. The factors 3, 7 occur in both cases: their product is therefore the G.C.M. of 420 and 1617.

We will in conclusion prove two important theorems relating to prime numbers. The first is this:

In whatever mode a number be resolved into its prime factors the final result will always be the same.

For instance, we may say 617 is the product of 3 and 539; 539 is the product of 7 and 77. 77 is the product of 7 and 11; so that 1617 is the product of 3, 7, 7, 11. But we may say instead 1617 is the product of 11 and 147; 147 is the product of 3 and 49; 49 is the product of 7 and 7; so that 1617 is the product of 3, 7, 7, 11. The prime factors we obtain at last 3, 7, 7, 11 will be the same whichever way we proceed. The prime numbers are the same, and each occurs the same number of times in both cases.

If a number could be resolved into prime factors in two different ways we should have the products formed by multiplying two different sets of prime numbers equal. Divide by the factors that are common to both sets; the products of the numbers that are left will be equal. Take any one of these numbers; it obviously divides one product and must therefore divide the other product. But this is impossible, for it is prime to each of the factors entering into the product and therefore it is prime to the product itself. Hence the products formed by multiplying two different sets of prime numbers cannot be equal, and a number cannot be resolved into prime factors in two different ways.

We may repeat the reasoning with actual numbers. Without actually multiplying we may shew that the product of the numbers 2, 3, 3, 3, 5, 5, 7, 7, 11 cannot be equal to the product of the numbers 2, 2, 3, 5, 5, 5, 5, 11,

11. For if these two products were equal we should have, dividing by the common factors 2, 3, 5, 5, 11, the product of the numbers 3, 3, 7, 7 equal to the product of the numbers 2, 5, 5, 11. But this is impossible, for 3 divides the first product and cannot divide the second since it is prime to each of the factors 2, 5, 5, 11.

The second theorem is this :

However great a prime number be given it is always possible to find one greater.

It may be stated in simpler words thus:

The series of prime numbers is unlimited.

Or thus:

There is no last prime number.

To prove this we take all the prime numbers up to the given prime number and multiply them together. We then add one to the result. We thus obtain a number not divisible by any of the prime numbers we have taken since division by any of these leaves a remainder 1. This number must then either be a prime number or else can be resolved into prime factors which will not belong to the series of prime numbers written down. In either case we obtain a prime number greater than the given prime number.

SECTION 4.

Least Common Multiple.

A common multiple of several numbers is a number which is divisible by all these numbers.

The least common multiple of several numbers is the least number which is divisible by all these numbers.

For instance 60, 120, 180, 240 are common multiples of 12 and 15, and 60 is the least common multiple of 12 and 15.

To find the L.C.M.* of two numbers we may resolve each number into prime factors and then multiply all the prime factors together, not repeating however the factors which occur in both numbers.

Thus the prime factors of 12 are 2, 2, 3, and those of 15 are 3, 5, and to find the L.C.M. of 12 and 15 we multiply together 2, 2, 3, 5, writing only once the 3 which is a factor of both numbers.

To find the L.C.M. of 420 and 1617 we resolve 420 into its prime factors 2, 2, 3, 5, 7 and 1617 into its prime factors 3, 7, 7, 11. We then multiply together the numbers 2, 2, 3, 5, 7, 7, 11, not repeating the factors 3, 7, when we come to write down the factors of the second numbers because they have already been written among the factors of the first. The L.C.M. of 420 and 1617 is thus found to be 32340.

Before proving the method just given we may remark that it is equivalent to multiplying all the factors 2, 2, 3, 5, 7, 3, 7, 7, 11, and then dividing by the factors 3, 7 common to both numbers; that is to say it is equivalent to multiplying the two numbers together and dividing the product by their greatest common measure. The L.C.M. of two numbers is then the result of dividing

* L.C.M. is an abbreviation for least common multiple.

their product by their G.C.M., or in other words the product of the L.C.M. and the G.C.M. of two numbers is equal to the product of the number themselves. In this case the product of 32340 and 21 is equal to the product of 420 and 1617.

Now it is obvious that the product of all the factors 2, 2, 3, 5, 7, 7, 11, is a multiple of the product of the factors 2, 2, 3, 5, 7 and also of the product of the factors 3, 7, 7, 11. 32340 is then *a* common multiple of 420 and 1617, but how do we know it is the *least* common multiple?

To shew this, we must prove the following theorem:

If a number be divisible by each of two numbers prime to one another it is also divisible by their product.

Thus 8 and 15 both divide 2280, and we have to prove that 2280 is also divisible by their product 120. Since 8 divides 2280 the G.C.M. of 8 and 2280 is 8 itself. Now we saw in the last section that before finding the G.C.M. of two numbers we may divide either of these by any factor prime to the other. But 15 is prime to 8; hence dividing 2280 by 15 we see that the G.C.M. of 8 and 152 is 8 itself. 8 then divides 152, and therefore 15 times 8 divides 15 times 152.

Or we may repeat the proof before given to shew that 8 being prime to 15 and dividing 15 times 152, it must divide 152 itself.

8 obviously divides 8 times 152; since then it divides 15 times 152 it must divide the difference 7 times 152. Again since 8 divides 7 times 152 and 8 times 152 it will divide their difference 152 itself.

Having proved then that *any* common multiple of 8 and 15 is also a multiple of 120 their product, it is clear that the *least* common multiple of 8 and 15 will be that very product 120. Take now a third number 7 for example, prime to each of the preceding numbers, then we may shew that any number divisible by 8, 15 and 7 is divisible by their product 840. For any number divisible by 8 and 15 is divisible by 120, and since 8 and 15 are both prime to 7 their product 120 is prime to 7, so that any number divisible by 120 and 7 is divisible by 840, the product of these numbers. Hence the least common multiple of the three numbers 8, 15, 7 is their product 840. By proceeding in this way we may shew that however many numbers we take, if these be all prime to one another, their least common multiple will be their product.

We will now return to the numbers 420 and 1617. Since 420 is the product of 4, 3, 5, 7 any number divisible by 420 must be divisible by each of the numbers 4, 3, 5; and since 1617 is the product of 3, 49, 11 any number divisible by 1617 must be divisible by each of the numbers 49, 11. It follows then that any number divisible by both 420 and 1617 must be divisible by all the numbers 4, 3, 5, 49, 11. Now these numbers are all prime to one another, and therefore the least number divisible by all of them is their product 32340. No common multiple of 420 and 1617 can be less than 32340, and we have already seen that 32340 is a common multiple of 420 and 1617. Hence 32340 is the least common multiple of 420 and 1617.

As another example we will find the L.C.M. of 320 and 700.

The factors of 320 are 64 and 5, and the factors of 700 are 7, 4 and 25. We reject the 5 since it is a factor of 25 and the 4 since it is a factor of 64. We multiply together all the other factors, namely 64, 7 and 25. The number so obtained is clearly divisible both by the product of 64 and 5, and by the product of 4, 7, and 25. Again, since 64, 7 and 25 are prime to one another the least number divisible by all of these numbers is their product 11200. Hence 11200 is the required L.C.M. of 320 and 700.

We see once more that the L.C.M. of two numbers is equal to the product of the numbers divided by their G.C.M. For multiplying the factors 64, 7, 25 together is equivalent to multiplying all the factors of both numbers together, namely 64, 5, 4, 25 and then dividing by the factors 5, 4. But since these factors are common to the two numbers their product 20 is the G.C.M. in question.

When we wish to find the L.C.M. of several numbers we may take any two of them and find their L.C.M., then combine the result with a third number and find *their* L.C.M. and so on. A more convenient method in practice is that indicated below.

2	6,	14,	18,	24,	30,	56
3	3	7	9	12	15	28
7	1	7	3	4	5	28
4	1	1	3	4	5	4
	1	1	3	1	5	1

Here we wish to find the L.C.M. of 6, 14, 18, 24, 30, 56. We write the numbers down in a row and divide them by any numbers which will divide more than one.

In this case we begin with 2 and write down the quotients in the line beneath. We next divide as many numbers as possible by 3, and those numbers such as 7 and 28 which cannot be divided by 3 we write down unaltered. We continue the process till the numbers left are all prime to one another. The divisors must be either actually prime like 2, 3, 7, or else like 4, prime to those numbers 3, 5 which they do not divide exactly. The product of all the divisors and the numbers in the last row is the L.C.M. required. In this case the L.C.M. is the product of 2, 3, 7, 4, 3, 5 or 2520. The reader, from what has been said concerning the L.C.M. of two numbers, will find no difficulty in supplying the proof of the rule.

Theorem Relating to Whole Numbers continued.

In this section we shall only give one or two simple theorems concerning whole numbers which have not yet been mentioned.

If we add all the divisors of a number including unity but excluding the number itself the sum will in certain cases be equal to the original number. In these cases the number was called by the Greeks "perfect." Thus 6 is a perfect number because the divisors of 6 are 1, 2 and 3, and 6 is equal to the sum of 1, 2 and 3.

Euclid, in the last proposition of the 9th book of the "Elements" gives a simple rule for finding perfect numbers. Take any number of terms of the series 1, 2, 4, 8, 16,... where each term is double the last and add them all together. The sum will in certain cases be a prime number. When this is so, multiply the sum by the last term and we shall have a perfect number.

For instance the sum of 1 and 2 is 3, a prime number. Multiply this sum 3 by 2 and we have 6 a perfect number.

The sum of 1, 2 and 4 is 7, a prime number. Multiply this sum 7 by 4 and we have 28, a perfect number. In fact 28 is the sum of its divisors 1, 2, 4, 7, 14.

The sum of 1, 2, 4 and 8 is 15, which is not a prime number.

The sum of 1, 2, 4, 8, 16 is 31, a prime number. Multiply 31 by 16 and we have 496, a perfect number.

When the sum of the divisors was greater than the number itself, the number was considered to have too many divisors and was called "excessive." If the sum of the divisors was less than the number itself the number was called "defective." Thus 12 is an excessive number since the sum of its divisors 1, 2, 3, 4, 6 is 16 and 8 is a defective number since the sum of its divisors 1, 2, 4 is 7.

Numbers so related that each is equal to the sum of the divisors of the other were called "friendly" or "amicable."

Thus 220 and 284 are two amicable numbers since the divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 and their sum is 284, while the divisors of 284 are 1, 2, 4, 71, 142 and their sum is 220.

These definitions are only of historical interest. The following theorem is however of more importance.

If we take as many consecutive numbers as we please beginning with any number, the product of these numbers will always be divisible by the product of the same number of consecutive numbers beginning with one.

Thus the product of any four consecutive numbers such as 7, 8, 9, 10 is divisible by the product of the numbers 1, 2, 3, 4.

The theorem is obvious for two numbers. In fact one of the numbers must be even, and hence their product will be even also. If we have three consecutive numbers it is easy to see that one of them must be divisible by 3. Hence the product of the numbers will be divisible by 3. Again, one at least of the three numbers is divisible by 2 and therefore their product is divisible by 2. The product then of the three numbers is divisible by both 2 and 3, and is therefore since 2 and 3 are prime to one another divisible by their product 6.

We cannot continue the reasoning in exactly the same way for four numbers since 2 and 4 are not prime to one another, and it is not enough to shew that the product of the four numbers is divisible by 2, 3, 4 separately. We may however reason as follows. Suppose 7, 8, 9, 10 are the four numbers. Now 10 times the product of 7, 8, 9 is equal to 4 times the product of 7, 8, 9, together with 6 times the product of 7, 8, 9. We have however seen that the product of 7, 8, 9 is divisible by the product of 1, 2, 3. And therefore 4 times the product of 7, 8, 9 will be divisible by the product of 1, 2, 3, 4. If then the product of 6, 7, 8, 9 is divisible by the product of 1, 2, 3, 4 the product of 7, 8, 9, 10 will also be divisible by the product of 1, 2, 3, 4. Repeating the same reasoning we see that the product 6, 7, 8, 9 is divisible by the product of 1, 2, 3, 4 if the product of 5, 6, 7, 8 is. The product of 5, 6, 7, 8 is divisible if the product of 4, 5, 6, 7 is. Now since the product of 5, 6, 7 is divisible by the product of 1, 2, 3, the product of 4, 5, 6, 7 is divisible by the product of 1, 2, 3, 4. It follows then retracing our steps that the product of 7, 8, 9, 10 is divisible by the product of 1, 2, 3, 4, and so also is the product of any other four consecutive numbers.

For five numbers we may proceed in the same way. The product of 12, 13, 14, 15, 16 is equal to 5 times the product of 12, 13, 14, 15 together with 11 times the product of 12, 13, 14, 15. Now the product of 12, 13, 14, 15 is divisible by the product of 1, 2, 3, 4 and therefore 5 times the product of 12, 13, 14, 15 is divisible by the product of 1, 2, 3, 4, 5. All then that is required is to prove that the product of 11, 12, 13, 14, 15 is divisible by the product of 1, 2, 3, 4, 5. Repeating the reasoning we arrive successively at the following sets of numbers, 10, 11, 12, 13, 14; 9, 10, 11, 12, 13; 8, 9, 10, 11, 12; 7, 8, 9, 10, 11; 6, 7, 8, 9, 10; 5, 6, 7, 8, 9. But the product of 5, 6, 7, 8, 9 is divisible by the product of 1, 2, 3, 4, 5 since the product of 6, 7, 8, 9 is divisible by the product of 1, 2, 3, 4.

We may apply the same reasoning to 6, 7, &c. consecutive numbers.

Another way is to decompose the second product into factors prime to one another. Thus the product of 1, 2, 3, 4, 5, 6 is equal to the product of 2, 3, 4, 5, 3, 2, that is to the product of 9, 5, 16. We have then to prove that the product of any six consecutive numbers is divisible by the numbers 9, 5, 16. For since these numbers are prime to one another any number which is divisible by each of them separately will be divisible by their product. Now in any six consecutive numbers there will be two numbers divisible by 3, therefore the product of all six numbers must be divisible by 9. Again, in any six consecutive numbers there must be at least one number divisible by 5. Therefore the product of all six numbers must be divisible by 5. Lastly, one at least of the six numbers must be divisible by 4, and besides this number two others must be divisible by 2. Therefore the product of all six numbers is divisible by the product of 4, 2, 2, or by 16. The theorem is thus proved.

The following rule is useful in resolving a number into its prime factors. Unlike the previous theorems it depends on the fact that *ten* is the basis of our system of numeration.

Any number is divisible by 9 or 3 if the sum of its digits is divisible by 9 or 3, respectively.

Thus, in order to find whether 6972 is divisible by 9 or not we add the digits 6, 9, 7, 2 together and see whether their sum 24 is divisible by 9 or not. 24 is not divisible by 9, but it is divisible by 3, hence 6972 will not be divisible by 9, but will be divisible by 3.

In fact 6000 is equal to 6 times 999 together with 6,
900 is equal to 9 times 99 together with 9,
70 is equal to 7 times 9 together with 7.

Hence 6972 is equal to the sum of the numbers 6 times 999, 9 times 99, 7 times 9, 6, 9, 7, 2. But the first three numbers are clearly divisible by both 9 and 3, since 999, 99, 9 are divisible by these numbers. It follows that the whole number 6972 will be divisible by 9 or 3 if the sum of the digits 6, 9, 7, 2 is divisible by 9 or 3. Also it is clear that on dividing any number by 9 or 3 the remainder will be the same as if we divided the sum of its digits by 9 or 3.

EXAMPLES.

1. Prove that 6 times 9 is equal to 9 times 6.

2. Find the G.C.M. of 819 and 3588 and also the G.C.M. of 841 and 1624.

3. What are the prime factors of 8240?

4. Find the G.C.M. of 8240 and 3339.

5. Shew that in using the "Sieve of Eratosthenes" to find prime numbers we need only begin with 25 in crossing out numbers divisible by 5, with 49 in crossing out numbers divisible by 7, and with 121 in crossing out numbers divisible by 11.

6. Find by this method all the prime numbers up to 150.

7. Is 851 a prime number?

8. The number 7 is prime both to 12 and to 15, prove that it is prime to their product 180.

9. Find the L.C.M. of 819 and 3588.

10. Find the L.C.M of 35, 60, 42, 15, 24.

11. Verify the fact that 496 is a perfect number.

12. What is the next perfect number after 496?

13. Prove that any multiple of a perfect number such as 28 for example is an excessive number.

14. Prove that the following theorems result from the commutative and associative properties of addition.

(*a*) When one number is subtracted from another if the larger number be increased or diminished the remainder will be increased or diminished by the same amount.

(*b*) When one number is subtracted from another if the number to be subtracted be increased or diminished the remainder will be diminished or increased by the same amount.

(*c*) When one number is subtracted from another if both numbers be increased or diminished by the same amount the remainder will be unaltered.

15. Prove that the following theorems result from the commutative and associative properties of multiplication.

(*a*) When one number is divided by another if the dividend be increased or diminished any number of times, the quotient will be increased or diminished the same number of times.

(*b*) When one number is divided by another if the divisor be increased or diminished any number of times the quotient will be diminished or increased the same number of times.

(*c*) When one number is divided by another if both dividend and divisor be increased or diminished the same number of times the quotient will be unaltered.

16. Prove that the following theorems result from the distributive property of multiplication.

(*a*) When the difference of two numbers is multiplied by another number the product obtained is the difference of the products that would have been obtained if each of the former numbers had been multiplied separately by the latter number.

(*b*) The quotient obtained by dividing the sum of two numbers by another number is the sum of the quotients that would be obtained if each of the former numbers were divided separately by the latter number.

(*c*) The quotient obtained by dividing the difference of two numbers by another number is the difference of the quotients that would be obtained if each of the former numbers were divided separately by the latter number.

Note. In all these theorems it is supposed that the division can be performed exactly without remainder.

CHAPTER IV.

SECTION I.

FRACTIONS.

It is not generally possible to divide a number of distinct objects exactly into equal parts. In most cases there will be a remainder. But it is always possible to divide a continuous quantity such as a length, an area, or space of time into as many equal parts as we wish. Thus we cannot distribute 6 objects equally among 4 people, but we can divide a length of 6 inches into 4 equal parts. We call the length so obtained a fraction of an inch even though it be actually greater than an inch.

A fraction of any unit of measure (such as an inch, an hour, an acre) is the quantity obtained by dividing any multiple of the unit into any number of equal parts whenever that quantity is not in itself an exact multiple of the unit.

The length obtained by dividing 6 inches in 4 equal parts is expressed in terms of inches by the symbol $\frac{6}{4}$.

The original number of inches (in this case 6) is called the numerator; the number of parts (in this case 4) into which the whole quantity is divided is called the denominator.

We can add or subtract two portions of time whether they be exact multiples of the unit of time or not; and the same is true of two lengths, or of two quantities of any kind whatever. If now the two portions of time be expressed by means of fractions in terms of the unit of time we can endeavour to find what fraction will express the sum of the two portions or the remainder when one is subtracted from the other. We can thus add and subtract two fractions in the same sense as we add and subtract two wholes. And just as we can say 2 and 3 are 5 meaning 2 objects and 3 objects of any kind are equal to 5 objects of the same kind, so we can say $\frac{1}{2}$ and $\frac{1}{4}$ are $\frac{3}{4}$ meaning $\frac{1}{2}$ a unit and $\frac{1}{4}$ a unit of any kind are equal to $\frac{3}{4}$ a unit of the same unit.

The properties of fractions like those of whole numbers are independent of the special kind of unit. We may then in defining fractions drop all explicit mention of the unit of measure and say,

A fraction is the result of dividing one number by another whenever that result is not itself a whole number.

But it must be remembered that some unit of measure is always implied. Just as 2 has no meaning by itself apart from some kind of object so $\frac{6}{4}$ has no meaning apart from some unit of measure.

We may think of $\frac{6}{4}$ inches as meaning not only 6 inches divided by 4 but also 6 times $\frac{1}{4}$ of an inch. In fact six inches are equal to twenty-four quarters of an inch, and the fourth part of twenty-four quarters of an inch is six quarters of an inch. Or we may take a slip of paper six inches long and fold it on itself into six pieces of an inch each. Then by cutting through all the pieces at once we shall divide the whole length in 4 equal parts and at the same time our portions will consist of six pieces each a quarter of an inch. The reader will observe that this proof is really identical with that of the commutative theorem in multiplication, and it is easy to see that this property of fractions is a result of the commutative theorem. When we name fractions in words we think of the latter meaning. We express the symbol $\frac{6}{4}$ by the words six-fourths, and $\frac{3}{5}$ by the words three-fifths.

A fraction can be expressed in many different ways, and as we have seen already, fractions can be added and subtracted. By an extended definition of multiplication and division fractions can also be multiplied and divided.

We must therefore consider in order, first the reduction of fractions to their simplest form, and then the means of performing the different arithmetical operations on them.

SECTION II.

Reduction of Fractions.

We have already seen that when one number is divided by another if the dividend and divisor be both increased or diminished the same number of times the quotient is unaltered. Thus if we divide 30 by 6 the quotient is 5; double both dividend and divisor, we shall then have to divide 60 by 12 and the quotient will still be 5. (Chap. III. Ex. 15 *c*.) This property is also true for fractions, and may be stated thus :

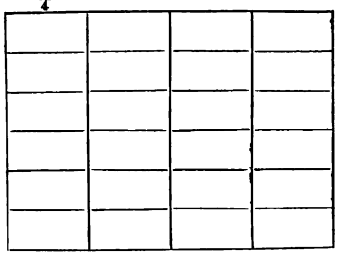
If the numerator and denominator be both increased or diminished the same number of times the fraction will be unaltered.

For instance $\frac{6}{4}$ is equal to $\frac{12}{8}$, $\frac{18}{12}$, $\frac{24}{16}$, &c., and is also equal to $\frac{3}{2}$.

To shew this suppose our unit of measure is an inch. Then by $\frac{6}{4}$ we mean a length which when multiplied by 4 will give a result equal to 6 inches. But this same length when multiplied by 8 will give a result double as great, that is to say, 12 inches. Now the length which when multiplied by 8 gives for result 12 inches is represented by $\frac{12}{8}$. Hence $\frac{6}{4}$ is equal to $\frac{12}{8}$. Again, the same length when multiplied by 12 will give a result three times as great as when multiplied by 4, that is to say, the result will be 18 inches. Hence $\frac{6}{4}$ is equal to $\frac{18}{12}$. Once more, the length which when multiplied by 4 gives six inches, when multiplied by 2 will give only half as much, namely 3 inches. Hence $\frac{6}{4}$ is equal to $\frac{3}{2}$.

We may illustrate the theorem in the following way. Let each of the horizontal lines be 6 units, and divide each of these lines into 4. Then each of these parts will

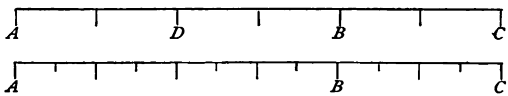
$\frac{6}{4}$



be $\frac{6}{4}$ of a unit. Now if there be 7 lines, for example, the whole number of lines will contain 7 times 6 or 42 units, and they will at the same time contain 7 times 4 or 28 parts. Hence each part is $\frac{42}{28}$ of a unit. Therefore $\frac{6}{4}$ is equal to $\frac{42}{28}$.

Or, to take another instance, let a week be our unit of time. 3 days is equal to the 7th part of 3 weeks, to the 14th part of 6 weeks, to the 21st part of 9 weeks, &c. Hence $\frac{3}{7}$, $\frac{6}{14}$, $\frac{9}{21}$, &c. are equal.

We saw that we might consider $\frac{6}{4}$ inches as meaning not only the 4th part of 6 inches, but also 6 times the 4th part of an inch. From this latter point of view the theorem is equally easy to prove. Let *AB* be an inch and divide it into 4 parts.



Let *AC* be 6 of these parts. Then *AC* is $\frac{6}{4}$ of an inch. Now divide each of the parts into two equal parts. Then there will be 8 of these smaller parts in *AB*, and *AC* will contain 12. Therefore *AC* is $\frac{12}{8}$ of an inch. Hence $\frac{6}{4}$ is equal to $\frac{12}{8}$.

Also if *AD* is the half of an inch *AC* will be three times *AD*, since *AD* is two quarters of an inch, and *AC* is six quarters of an inch. Therefore $\frac{3}{2}$ is equal to $\frac{6}{4}$.

Again, twice the fifth part of an hour is equal to twice twelve minutes or to 24 minutes.

But this is also equal to 4 times the 10th part of an hour, (4 times 6 minutes) to 6 times the 15th (6 times 4 minutes) and to 8 times the 20th part of an hour (8 times 3 minutes), so that the fractions $\frac{2}{5}$, $\frac{4}{10}$, $\frac{6}{15}$, $\frac{8}{20}$, are all equal.

Since we have seen that when the numerator and denominator of a fraction are divided by the same number the fraction is unaltered, we may always simplify a fraction when the numerator and denominator have common factors. If then a fraction is given us we enquire first of all what factors are common to both numerator and denominator. This may be done either by finding their G.C.M. or else by trying prime numbers in manner explained already. When we have either divided at once both numerator and denominator by their G.C.M. or else divided successively by all the common factors, so that now the numerator and denominator are prime to one another, the fraction is said to be in its lowest terms.

Take for example the fraction $\frac{84}{210}$. Halving both numerator and denominator the fraction is equal to $\frac{42}{105}$. After dividing both by 3 we find it equal to $\frac{14}{35}$. Lastly, after dividing by 7 we obtain $\frac{2}{5}$. Now 2 and 5 have no common factor or are prime to one another. Hence the fraction $\frac{2}{5}$ is said to be in its lowest terms, and we are said to have reduced the fraction $\frac{84}{210}$ to its lowest terms when we find the equivalent fraction $\frac{2}{5}$.

Again, take the fraction $\frac{85}{204}$. We find the G.C.M. of 85 and 204 thus:

```
85) 204 (2
    170
    ---
     34) 85 (2
         68
         --
         17) 34 (2
             34
             --
```

We divide both 85 and 204 by 17 and obtain for quotients 5 and 12.

Hence $\frac{85}{204}$ is equal to $\frac{5}{12}$, and since 5 and 12 are prime to one another, the fraction is reduced to its lowest terms.

If the numerator of a fraction be greater than the denominator we can express the fraction as the sum of a whole number and another fraction whose numerator is less than the denominator. Thus six quarters are equal to four quarters and two quarters, just as six books are equal to four books and two books. But four quarters are equal to one. Hence six quarters are equal to one and two quarters. Using symbols we say $\frac{6}{4}$ is equal to $1\frac{2}{4}$. The sum of 1 and $\frac{2}{4}$ is indicated by putting 1 close to the $\frac{2}{4}$.

Seventeen-sevenths is equal to fourteen-sevenths and three-sevenths, that is to say to two and three-sevenths. In symbols $\frac{17}{7}$ is equal to $2\frac{3}{7}$.

It follows, then, that every fraction can be expressed as a fraction in which the numerator is less than and prime to the denominator, or else as the sum of such a fraction and a whole number. In bringing the fraction to such a form we may either first make the numerator less than and afterwards prime to the denominator, or we may first make it prime to and afterwards less than the denominator. We may make $\frac{6}{4}$ equal first to $1\frac{2}{4}$ and then to $1\frac{1}{2}$, or else first to $\frac{3}{2}$ and after that to $1\frac{1}{2}$.

SECTION III.

Addition and Subtraction of Fractions.

When fractions have the same denominator we can add them or subtract them at once—$\frac{3}{7}$ and $\frac{2}{7}$ are together equal to $\frac{5}{7}$, and $\frac{2}{7}$ taken from $\frac{3}{7}$ leave $\frac{1}{7}$, just as 3 apples and 2 apples are together equal to 5 apples and 2 apples taken from 3 apples leave 1 apple. Here the addition takes place just as with whole numbers. The only difference is that instead of adding multiples of our original unit of measure which we may suppose a foot, we are adding multiples of a new unit of measure namely one seventh of a foot. We must reduce all other cases to this simple case. When we have two fractions to add we must discover a new unit of measure in terms of which they can both be expressed as whole numbers. For instance let us add $\frac{1}{3}$ and $\frac{1}{4}$ and let our unit of measure again be a foot. We have to ask, In terms of what unit of measure can $\frac{1}{3}$ of a foot and $\frac{1}{4}$th of a foot be expressed as whole numbers? The answer is clearly: In inches. $\frac{1}{3}$ of a foot is 4 inches and $\frac{1}{4}$ of a foot is 3 inches. 4 inches and 3 inches together make 7 inches, that is to say $\frac{7}{12}$ of a foot. Hence $\frac{1}{3}$ and $\frac{1}{4}$ are together equal to $\frac{7}{12}$.

We see that the reason why $\frac{1}{3}$ and $\frac{1}{4}$ of a foot can both be expressed as multiples of an inch is that 3 and 4 both divide 12 the number of inches in a foot. When then we have two fractions to add we must find a number divisible by both denominators, and divide the original unit by this number in order to obtain the new unit of measure. Suppose now we wish to add $\frac{5}{6}$ and $\frac{3}{8}$. 24 is divisible by both 6 and 8. We must then express both $\frac{5}{6}$ and $\frac{3}{8}$ in terms of the 24th part of the original unit.

The original unit will itself contain 24 such parts and therefore $\frac{1}{6}$th of the unit will contain 4 parts and $\frac{5}{6}$ths will contain 20 parts. So also $\frac{1}{8}$th of the original unit will contain 3 of the 24th parts and $\frac{3}{8}$ths will contain 9 parts. Hence $\frac{5}{6}$ and $\frac{3}{8}$ together contain 29 of the 24th parts, that is to say their sum is equal to $\frac{29}{24}$ or to $1\frac{5}{24}$.

Here we have found $\frac{5}{6}$ equal to $\frac{20}{24}$ and $\frac{3}{8}$ to $\frac{9}{24}$. We might have known this at once from the last section. For we saw there that a fraction was unaltered when its numerator and denominator were both multiplied by the same number. Now multiplying the numerator and denominator of $\frac{5}{6}$ by 4 we have $\frac{20}{24}$ and multiplying the numerator and denominator of $\frac{3}{8}$ by 3 we have $\frac{9}{24}$. The fractions have now the same denominator and can be added immediately.

Before then fractions can be added they must be made to have the same denominator, or in the language generally employed they must be reduced to a common denominator. This is equivalent to the former statement that they must be expressed as whole numbers in some new unit of measure. The common denominator must be a multiple of the original denominators of the two fractions, and it is usual to choose the least common multiple. The fraction we obtain on adding may then not need to be simplified. If in the present example we had chosen 48 as common denominator we should have found that $\frac{5}{6}$ was equal to $\frac{40}{48}$ and $\frac{3}{8}$ to $\frac{18}{48}$ and their sum to $\frac{58}{48}$. But now the fraction is not in its simplest form. 58 and 48 have a common factor 2 and we must divide both numerator and denominator of $\frac{58}{48}$ by this factor before we can obtain the former result $\frac{29}{24}$.

Several fractions can be added in the same manner as two. Let us add, $\frac{5}{8}$, $\frac{7}{12}$, $\frac{4}{15}$, $\frac{3}{20}$. We first of all find the L.C.M. of the denominator 8; 12, 15, 20.

4	8,	12,	15,	20
3	2,	3,	15,	5
5	2,	1,	5,	5
	2,	1,	1,	1

It is the product of the numbers 4, 3, 5, 2, that is to say 120.

We must now express all the fractions as fractions with 120 for denominator.

To make 120 the denominator of $\frac{5}{8}$, we multiply both numerator and denominator by 15 since 15 is the quotient found on dividing 120 by 8. We find that $\frac{5}{8}$ is equal to $\frac{75}{120}$. In the same way we find that $\frac{7}{12}$, $\frac{4}{15}$, $\frac{3}{20}$ are respectively equal to $\frac{70}{120}$, $\frac{32}{120}$, $\frac{18}{120}$. The whole sum is therefore $\frac{195}{120}$ which is equal to $\frac{13}{8}$ or to $1\frac{5}{8}$.

If the fractions to be added are greater than one it is best to express them as the sums of whole numbers and fractions less than 1, and then to add first the whole numbers and afterwards the fractions together.

Suppose we have to add $\frac{40}{9}$, $\frac{15}{7}$ and $\frac{8}{21}$.

$\frac{40}{9}$ is equal to 4 and $\frac{4}{9}$; $\frac{15}{7}$ is equal to $2\frac{1}{7}$. We add first 4 and 2 together and obtain 6. Next we add $\frac{4}{9}$, $\frac{1}{7}$ and $\frac{8}{21}$ together. The L.C.M. of 9, 7, 21 is 63. $\frac{4}{9}$ is equal to $\frac{28}{63}$, $\frac{1}{7}$ is equal to $\frac{9}{63}$, $\frac{8}{21}$ is equal to $\frac{24}{63}$. Therefore their sum is $\frac{61}{63}$ and the whole sum is $6\frac{61}{63}$.

In both these examples as the reader may easily verify we could have chosen any other common multiple instead of the least common multiple of the denominators, but we should have had to multiply by larger numbers in reducing the fractions to a common denominator and to divide by larger numbers in simplifying the result.

The Subtraction of fractions will now need no explanation. We reduce the fractions to a common denominator and then subtract the numerators. Suppose we wish to subtract $\frac{3}{8}$ from $\frac{7}{12}$. The L.C.M. of 8 and 12 is 24. $\frac{7}{12}$ is equal to $\frac{14}{24}$ and $\frac{3}{8}$ is equal to $\frac{9}{24}$. Hence the difference of $\frac{7}{12}$ and $\frac{3}{8}$ is $\frac{5}{24}$.

If whole numbers are combined with the fractions we first of all subtract the fractions and afterwards the whole numbers. Thus in subtracting $2\frac{5}{16}$ from $7\frac{11}{20}$ we first say $\frac{5}{16}$ from $\frac{11}{20}$ leaves $\frac{19}{80}$ and then 2 from 7 leaves 5; therefore the remainder is $5\frac{19}{80}$.

It will sometimes happen when there are whole numbers with the fractions that the fractional part (the part less than unity) of the smaller quantity is too great to be subtracted from the fractional part of the larger quantity. In that case we use an artifice similar to that already employed in the subtraction of whole numbers. We take one of the units of the larger quantity and combine with the fractional part to form a fraction greater than one. We can now subtract from this fraction the fractional part of the smaller quantity in the usual way. Let us subtract $2\frac{8}{9}$ from $4\frac{5}{12}$. $\frac{8}{9}$ is equal to $\frac{32}{36}$ and $\frac{5}{12}$ to $\frac{15}{36}$, and we cannot take $\frac{32}{36}$ from $\frac{15}{36}$. But $4\frac{15}{36}$ is equal to $3\frac{41}{36}$ and $\frac{32}{36}$ taken from $\frac{41}{36}$ leaves a remainder $\frac{9}{36}$ or $\frac{1}{4}$. Also 2 taken from 3 leaves a remainder 1, so that the answer is $1\frac{1}{4}$.

The reader will see that the principles in the Reduction, Addition and Subtraction of fractions are not new but are already involved in the operations on whole numbers. In fact whether in any concrete example we shall have to use fractions or whole numbers depends entirely on the arbitrary units of measure chosen. We can express the fifth part and the tenth part of a pound as 4*s.* and 2*s.* respectively, and their sum as 6*s.*, because the pound happens to have been divided into 20 smaller units of value and 20 is divisible by both 5 and 10. But the fifth and the tenth part of a foot can only be expressed by the fractions $\frac{1}{5}$ and $\frac{1}{10}$ and their sum by the fraction $\frac{6}{20}$ (or of course by some equivalent fractional form) since the number of smaller units into which a foot has been divided is not divisible by 5 or 10. Now in the first case no one will see anything but a question relating to whole numbers. The second case however merely differs from the first in the accidental fact that there is no special name for $\frac{1}{20}$th of a foot. The difference then is merely a difference in language. In any special case the use of fractions might have been avoided if the relations between the different units of measure (inch, foot, yard) had been suitably chosen. But of course these relations cannot possibly suit all cases, however chosen, so that we must generally ourselves make a smaller unit suited to the question in hand, and this is the whole artifice required for the addition and subtraction of fractions.

SECTION IV.

MULTIPLICATION AND DIVISION OF FRACTIONS.

The definition of multiplication already given will apply to the multiplication of a fraction by a whole number. To multiply a fraction by a whole number is to add the fraction to itself as many times as there are units in the number. This addition is especially easy since the fractions have necessarily the same denominator. We have then merely to add the numerators, or what is the same thing, to multiply the numerator of the fraction by the whole number. Thus multiplying $\frac{4}{7}$ by 5, we obtain $\frac{20}{7}$, just as multiplying 4 books, 4 houses, &c. by 5 we obtain 20 books, 20 houses.

The rule then for multiplying a fraction by a whole number is to multiply the numerator by the whole number and leave the denominator unaltered.

We saw that there were two ways of looking at division. To divide 20 by 4 may either mean to find how many books each person will have when 20 books are divided equally among 4 persons, or else it may mean to find how many times we can take 4 books away from

20 books. If we adopt the first form of question we can divide any fraction by a whole number.

For example we wish to divide $\frac{4}{7}$ by 5. We ask the following question: If $\frac{4}{7}$ of an inch is divided into 5 equal parts, what fraction of an inch is each part? Here we have first to divide 4 inches into 7 equal parts and then to divide each of these parts again into 5 smaller parts. We thus divide the whole 4 inches into 35 of the smaller parts, and each of the smaller parts will be $\frac{4}{35}$ of an inch. The result of dividing $\frac{4}{7}$ by 5 is therefore $\frac{4}{35}$.

The rule then for dividing a fraction by a whole number is to multiply the denominator by the whole number and leave the numerator unaltered.

If we multiply $\frac{4}{7}$ by 5, we obtain $\frac{20}{7}$, and if we reverse the operation and divide $\frac{20}{7}$ by 5, we obtain $\frac{20}{35}$, so that $\frac{20}{35}$ must be equal to the original fraction $\frac{4}{7}$. This agrees with the theorem that a fraction is unaltered when its numerator and denominator are multiplied by the same number.

But we cannot in the present case adopt the second form of question. The words "How many times can we take 5 inches away from $\frac{4}{7}$ of an inch?" have no meaning. There are cases however in which the second form will have a meaning and not the first. For example, we wish to divide $\frac{3}{4}$ by $\frac{1}{8}$. We cannot say, "What portion of an inch is there in each part when $\frac{3}{4}$ of an inch is divided into $\frac{1}{8}$ equal parts?" but we can say, "How many times is it possible to take $\frac{1}{8}$ of an inch away from $\frac{3}{4}$ of an inch?" In fact twice the 8th part of an inch will be the quarter of an inch, and 6 times the 8th part of an inch will be three-quarters of an inch. So that the answer is 6. We see then that the first form of question can be put whenever the divisor is a whole number, and the second form whenever the quotient is a whole number.

Hitherto we have adhered to the original definitions of multiplication and division. And we have seen that it is only in certain cases that these definitions have any meaning as applied to fractions. If then we are to multiply and divide fractions in general we must give new definitions of the words "multiplication" and "division." We begin with multiplication. What meaning, for example, are we to attach to the multiplication of $\frac{7}{8}$ by $\frac{5}{6}$? We must choose our definition so that the properties of multiplication may still if possible be the same, or else there would be no reason for continuing to use the same word.

Now one of the fundamental theorems relating to multiplication is, as we have seen, the Associative Theorem (Chap. III. Sec. I. Theorem V.). This theorem asserts that when the multiplier is increased any number of times (the multiplicand remaining unaltered) the product will be increased the same number of times. For instance, we multiply 9 by 4 and the product is 36; we now double the multiplier 4 and multiply 9 by 8, the product 36 will be doubled and become 72. We will make this theorem if possible hold true for fractions also. We will choose such a fraction for the product of $\frac{7}{8}$ and $\frac{5}{6}$ that when the multiplier $\frac{5}{6}$ is increased 6 times, the product itself will be increased 6 times also. Now 6 times $\frac{5}{6}$ is the whole number 5, and therefore 5 times the fraction $\frac{7}{8}$ will be

6 times the required product. But 5 times $\frac{7}{8}$ is equal to $\frac{35}{8}$, and the 6th part of $\frac{35}{8}$ is $\frac{35}{48}$. Hence the product of $\frac{7}{8}$ and $\frac{5}{6}$ must be chosen equal to $\frac{35}{48}$.

We arrive then at the following rule for the multiplication of two fractions.

Multiply the numerators together to form the new numerator, and the denominators together to form the new denominator.

We must now see whether the multiplication of fractions so defined possesses the same properties as the multiplication of whole numbers.

First of all we have seen that a fraction may be written in several forms without being altered. We must therefore shew that we obtain the same product when we apply the rule to the fraction in each of its forms. This is easy to prove. Suppose we write $\frac{20}{24}$ in the place of $\frac{5}{6}$. We shall now have to multiply 7 by 20 instead of by 5 in order to obtain the numerator of the product; so that (according to the associative theorem) the numerator will be 4 times 35 or 140 instead of 35. Similarly to find the denominator of the product we must multiply 8 by 24 instead of by 6, so that the denominator will be 4 times 48 or 192 instead of 48. But $\frac{140}{192}$ is the same as $\frac{35}{48}$ since we can divide both numerator and denominator by 4.

It is now easy to prove the Distributive Theorem for the multiplication of fractions. We will shew that multiplying the sum of $\frac{3}{10}$ and $\frac{7}{8}$ by $\frac{5}{6}$ gives the same result as multiplying $\frac{3}{10}$ by $\frac{5}{6}$ and $\frac{7}{8}$ by $\frac{5}{6}$ separately and then adding the products. We may reduce the fractions $\frac{3}{10}$ and $\frac{7}{8}$ to the same denominator. They thus become $\frac{12}{40}$ and $\frac{35}{40}$.

Now if we add these fractions, the numerator of the sum will be the sum of 12 and 35, and the denominator will be 40. Multiplying the fraction thus obtained by $\frac{5}{6}$ the numerator of the product will be 5 times the sum of 12 and 35, and the denominator will be 6 times 40. But this fraction is the sum of two fractions whose numerators are respectively 5 times 12 and 5 times 35, and whose denominators are both 6 times 40. But these fractions are the results of multiplying the fractions $\frac{12}{40}$ and $\frac{35}{40}$ respectively by the fractions $\frac{5}{6}$. The theorem is thus proved.

The commutative theorem is obviously true. When we multiply $\frac{7}{8}$ by $\frac{5}{6}$ we must multiply 7 by 5 to form the numerator, and 8 by 6 to form the denominator of the new fraction. If we multiply $\frac{5}{6}$ by $\frac{7}{8}$, the numerator will be 5 multiplied by 7, and the denominator 6 multiplied by 8. The results are the same in both cases, owing to the commutative theorem for whole numbers.

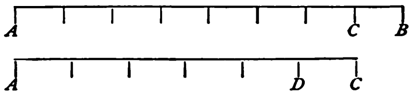
In like manner the associative theorem follows from the associative theorem for whole numbers.

Multiplication as applied to fractions possesses therefore all the properties of multiplication as applied to whole numbers. Now in Algebra, as the reader will learn, we are not concerned with the value of the symbols employed but solely with the relations between them.

Algebraically then there is no distinction whatever between multiplication of factors and multiplication of whole numbers. Arithmetically there is a very real distinction. Multiplication by a whole number (except one) always increases the quantity multiplied, while multiplication by a fraction often diminishes it. So that when we speak of multiplication as applied to fractions we are anticipating the algebraical point of view. We might dispense with the term as far as regards arithmetic alone.

We will now see what concrete meaning we can give to the multiplication of fractions.

Let an inch be our unit of measure, and let it be represented by *AB*. Divide *AB* in 8 equal parts, and take 7



of those parts to make the line *AC*. *AC* will be $\frac{7}{8}$ of an inch. Now divide *AC* into 6 equal parts, and take 5 of these parts to make *AD*. Each of the parts into which *AC* is divided is the 6th part of $\frac{7}{8}$ of an inch or $\frac{7}{48}$ of an inch. *AD* is equal to 5 such parts, and is therefore $\frac{35}{48}$ of an inch. Hence *AD* represents the result of multiplying $\frac{7}{8}$ of an inch by $\frac{5}{6}$. To multiply any quantity by $\frac{5}{6}$ is the same then as taking $\frac{5}{6}$ths of that quantity. The quantity becomes a new unit, and the length which would be expressed by $\frac{5}{6}$ in terms of that unit is expressed by $\frac{7}{8}$ multiplied by $\frac{5}{6}$ in terms of the old unit. From this point of view the distributive theorem is easily proved.

We now come to the division of fractions. We saw that it was a property of division of whole numbers that when the divisor is increased or diminished any number of times—the dividend being unaltered—the quotient is diminished or increased the same number of times. If we divide any number by 10 the result is half as great as if we had divided that number by 5; if we divide any number by 4 the result is three times as great as if we had divided that number by 12. We assume that this property holds true for fractions. This is no new assumption but is only the associative principle in another form. Suppose now we wish to divide $\frac{7}{11}$ by $\frac{3}{4}$. Since $\frac{3}{4}$ is the 4th part of 3, the result of dividing $\frac{7}{11}$ by $\frac{3}{4}$ must be 4 times as great as the result of dividing $\frac{7}{11}$ by 3. But we know that the result of dividing $\frac{7}{11}$ by 3 is $\frac{7}{33}$. Hence the result of dividing $\frac{7}{11}$ by $\frac{3}{4}$ must be $\frac{28}{33}$.

The following then is the rule for dividing one fraction by another.

Multiply the numerator of the first fraction (the dividend) by the denominator of the second to form the new numerator, and multiply the denominator of the first fraction by the numerator of the second to form the new denominator.

If we multiply $\frac{28}{33}$ by $\frac{3}{4}$ we obtain $\frac{84}{132}$. On dividing both numerator and denominator by 12 this is seen to be equal to $\frac{7}{11}$, the original fraction, as it ought to be.

If we have to multiply by a whole number and a fraction, such as $2\frac{3}{5}$, we may either multiply first by 2 and then by $\frac{3}{11}$, and afterwards add the results, or else we may put $2\frac{3}{5}$ in the form $\frac{13}{5}$ and apply the rule.

Before we can divide by $2\frac{3}{5}$ we must put it in the form $\frac{13}{5}$.

SECTION V.

Ratio and Proportion.

We now come to a new conception; the ratio of one quantity to another. Ratio cannot any more than number be defined in terms of other ideas; all we can do is to define what is meant by the quantity of two ratios. Since a ratio involves two quantities, in order to compare two ratios generally we shall require four quantities. We can give then the following definition of equality of ratio.

If we have four quantities, the first quantity is said to stand to the second in the same ratio as the third to the fourth when the first contains as many of the same kind of parts of the second as the third does of the fourth.

Quantities so related are said to be in proportion.

By "the same kind of parts" we mean that if the parts of the second quantity are fifth parts, the parts of the fourth quantity must be fifth parts also; if the parts of the second quantity are sixth parts, the parts of the fourth quantity must be sixth parts, &c.

Thus if the first quantity contains 3 times the 7th part of the second, the third must contain 3 times the 7th part of the fourth.

In this case if the second quantity were the unit of measure the first quantity would be expressed by the fraction $\frac{3}{7}$, and if the fourth quantity were the unit of measure the third quantity would be expressed by the fraction $\frac{3}{7}$. This will be true in all other cases, so we may say: Four quantities are in proportion when the first can be expressed in terms of the second as unit, and the third in terms of the fourth as unit by means of the same fractions.

We must include in our definition of equality of ratios two extreme cases; the first is when the first quantity is an exact multiple of the second, and the third is the same multiple of the fourth; for instance, the first quantity may be 6 times the second, and the third quantity 6 times the fourth; the second is when the first is contained an exact number of times in the second, and the third the same number of times in the fourth; for instance, the first may be the 9th part of the second, and the third the 9th part of the fourth. These cases will be included in the definition if we understand "as many" to mean, as many, whether one or more, and if we understand "part of" to mean either a portion of or else the whole.

We require two numbers in general to express a ratio, one to indicate the number of parts the first quantity contains, the other to indicate the number of parts into which the second quantity is divided. These numbers are written in succession with the arbitrary sign : between them. With the numbers just used the ratio would be expressed by the symbol 3 : 7.

Numbers like concrete quantities can have a ratio to one another. The numbers 3 and 7 stand the one to the other in the ratio 3 : 7, since 3 obviously contains 3 times the 7th part of 7. Hence in using the symbol 3 : 7 (which we read the ratio of 3 to 7) we are comparing all other equal ratios with one special ratio, the ratio of 3 to 7; just as in using the symbol 6 for the number of objects in a group we are comparing the group with one special group containing an equal number of objects, namely, the group 1, 2, 3, 4, 5, 6.

The ratio of two numbers such as 72 and 168 can be expressed in a simpler form than 72 : 168. For the 7th part of 168 is 24, and 72 is equal to 3 times 24. Hence 72 contains 3 times the 7th part of 168, and 3 : 7 is the ratio of 72 to 168. That is to say 3 : 7 is equal to 72 : 168. In general, when two numbers are both divided or multiplied by the same number, their ratio will be unaltered. If we divide the two numbers by their G.C.M. we can express the ratio in its simplest form as the ratio of two numbers prime to one another.

When four numbers are in proportion so that the first is to the second as the third to the fourth, the product of the first and fourth is equal to the product of the second and third. Take the numbers 10, 25, 6, 15, where the ratio of 10 to 25 is equal to the ratio of 6 to 15, then the product of 10 and 15 will be equal to the product of 25 and 6. For the ratio of 10 to 25 is equal to the ratio of 15 times 10 to 15 times 25, and the ratio of 6 to 15 is equal to the ratio of 6 times 25 to 15 times 25. The numbers 15 times 10 and 6 times 25 have therefore the same ratio to 15 times 25, and hence they must be equal. The relation between the numbers 10, 25, 6, 15 is often written in the form 10 : 25 :: 6 : 15, and expressed in words thus—ten is to twenty-five as six is to fifteen. The numbers 25, 6 standing in the middle are called the means, and the numbers 10, 15 standing at either end are called the extremes. Of course we can equally well write the proportion in the form 6 : 15 :: 10 : 25, so that 6 and 25 are the extremes, and 15 and 10 the means. But whichever way it is written the theorem just proved can be stated thus—In any proportion the product of the means is equal to the product of the extremes.

Since the product of 10 and 15 is equal to the product of 25 and 6 they both have the same ratio to the product of 6 and 15. But 10 times 15 is to 6 times 15 in the ratio of 10 to 6, and 25 times 6 is to 15 times 6 in the ratio of 25 to 15. It follows that the ratio of 10 to 6 is equal to the ratio of 25 to 15, or 10 : 6 :: 25 : 15. We have then the general theorem—If four quantities are in proportion so that the first is to the second as the third to the fourth, then the first will be to the third as the second to the fourth.

If the ratio of two numbers 221 and 260 be equal to the ratio of two numbers prime to one another 17 and 20, then 221 must be a multiple of 17 and 260 an equal multiple of 20. For we have just seen that 221 times 20 must be equal to 17 times 260. Hence 17 divides the product of 221 times 20; but it is prime to 20, therefore it must divide 221. We find the quotient to be 13, therefore 13 times 20 is equal to 260, or 20 divides 260 with the same quotient 13.

We can see now in another way that the second and third numbers in a proportion can be interchanged. Take the proportion 10 : 25 :: 6 : 15 once more. Dividing by the common factor 5 we can express the ratio 10 : 25 in the form 2 : 5, where 2 and 5 are prime to one another. Since then the ratio of 6 to 15 is equal to the ratio of 2 to 5 we must have 6 a multiple of 2, and 15 a multiple of 5. In fact 6 is 3 times 2, and 15 is 3 times 5. Knowing then that 10 is 5 times 2 and 6 is 3 times 2, we see that the ratio of 10 to 6 is equal to the ratio of 5 to 3; and knowing that 25 is 5 times 5 and 15 is 3 times 5, we see that the ratio of 25 to 15 is equal to the ratio of 5 to 3. Hence the ratio of 10 to 6 is equal to the ratio of 25 to 15*. It is clear also that the ratio of 25 to 10 will be equal to that of 15 to 6. In fact 25 contains 5 times the half of 10 and 15 contains 5 times the half of 6. Thus the fact that the numbers 10, 25, 6, 15 are proportional can be written in the following ways:

10 : 25 :: 6 : 15
6 : 15 :: 10 : 25
25 : 10 :: 15 : 6
15 : 6 :: 25 : 10
10 : 6 :: 25 : 15
25 : 15 :: 10 : 6
6 : 10 :: 15 : 25
15 : 25 :: 6 : 10

* This is—with the exception of some verbal alterations—the proof given in Euclid, *Elements* VII. 9, 10.

It is easy to find the ratio of one fraction to another fraction. Let us for example find the ratio of $\frac{3}{4}$ to $\frac{2}{3}$. We will suppose our unit to be a foot; $\frac{3}{4}$ of a foot is 9 inches, and $\frac{2}{3}$ of a foot is 8 inches. Therefore $\frac{3}{4}$ is 9 times the 8th part of $\frac{2}{3}$ or the ratio of $\frac{3}{4}$: $\frac{2}{3}$ is equal to the ratio of 9 : 8. Again, take any other fractions $\frac{5}{12}$ and $\frac{9}{14}$ for example. We reduce them to a common denominator, $\frac{5}{12}$ is equal to $\frac{70}{168}$; $\frac{9}{14}$ is equal to $\frac{81}{168}$. Hence $\frac{5}{12}$ contains 70 times the 81st part of $\frac{9}{14}$ and the ratio of $\frac{5}{12}$ to $\frac{9}{14}$ is 70 : 81.

The following then is the rule for finding the ratio of two fractions:

Reduce the fractions to a common denominator, the ratio of the numerators will be the required ratio.

A particular case of equal ratios is, as we have seen, the case when one of the first two numbers is an exact multiple of the other. When such a ratio is reduced to its simplest form one of the numbers expressing it will be unity. For instance, the ratio 12 : 4 is equal to the ratio 3 : 1 since 12 is the same multiple of 4 that 3 is of 1, and the ratio 2 : 10 is equal to the ratio 1 : 5 since 2 is contained in 10 the same number of times that 1 is in 5.

If four quantities are in proportion so that the ratio of the first to the second is equal to the ratio of the third to the fourth, then each of these ratios will be equal to the ratio of the sum of the first and second to the sum of the third and fourth. Take the numbers 12, 28, 18, 42. 12 contains 3 times the 7th part of 28 and 18 contains 3 times the 7th part of 42, so that the ratio of 12 to 28 is equal to the ratio of 18 to 42. All we have to shew then is that the sum of 12 and 18 is equal to three times the 7th part of the sum of 28 and 42. But this is obvious, for the 7th part of 28 and 42 is equal to the sum of 4 and 6, and 3 times the sum of 4 and 6 is equal to the sum of 12 and 18. Similarly we can shew that the difference between 18 and 12 is to the difference between 42 and 28 in the ratio of 12 to 28, or 18 to 42. These propositions are, as the reader will see, consequences of the distributive property of multiplication.

We have considered a ratio as having only the properties of equality or inequality with other ratios. If we consider a fraction from this point of view we shall find that it is identical with a ratio. The ratio 2 : 3 is identical with the fraction $\frac{2}{3}$ if we consider that fraction apart from any special unit of measure and solely in reference to its properties of equality or inequality with other fractions. Also we may identify the ratio 2 : 3 with the result of dividing 2 by 3, for the fraction $\frac{2}{3}$ is equal to 2 divided by 3 whatever the unit of measure may be. This gives us a means of extending the definition of division. We may define the quotient of one quantity by another by making it equal to the ratio of the first quantity to the second. This definition will apply to whole numbers and to fractions alike, and to concrete quantities, lengths, portions of time &c. even when they have not been expressed numerically. We can now find the result of dividing one fraction by another. To divide $\frac{5}{12}$ by $\frac{9}{14}$ we find the ratio of $\frac{5}{12}$ to $\frac{9}{14}$; and we may express the result, which we have already found to be 70 : 81 in the form $\frac{70}{81}$. Hence the rule for dividing one fraction by another is to reduce the fractions to a common denominator, and to divide one numerator by the other. This rule is identical in its results with the one formerly given.

Though the present method of looking at division seems to be distinct from that adopted in the last section, it is not so in reality. The reader will find that when we identify the result of dividing one quantity by another with the ratio of one quantity to another we tacitly assume the associative principle.

EXAMPLES.

1. Add $3\frac{7}{9}$, $6\frac{5}{12}$, and $\frac{3}{4}$ together.

2. Subtract $\frac{2}{9}$ from $\frac{7}{12}$.

3. Prove that a fraction less than 1 is increased by adding the same number to both numerator and denominator.

4. Reduce to its simplest terms $\frac{961}{899}$.

5. Which is greater, $\frac{355}{113}$ or $3\frac{1}{7}$, and what is the difference between them?

6. Add $\frac{1}{12}$, $\frac{1}{51}$, $\frac{1}{68}$.

7. Add $\frac{1}{42}$, $\frac{1}{86}$, $\frac{1}{129}$, $\frac{1}{301}$.

8. Add $\frac{1}{60}$, $\frac{1}{237}$, $\frac{1}{316}$, $\frac{1}{790}$.

9. Multiply the sum of $\frac{1}{16}$ and $\frac{1}{112}$ by the sum of 1, $\frac{1}{2}$, and $\frac{1}{4}$.

CHAPTER V.

SECTION I.

DECIMAL FRACTIONS.

THE labour of adding, subtracting, multiplying, dividing fractions is generally at least double that of adding, subtracting, multiplying or dividing, whole numbers. In multiplication, for instance, we must multiply the numerators together and the denominators, so that we have to work two ordinary multiplication sums. But there are certain cases when the operations of arithmetic are especially easy to perform on fractions. This will happen when the fractions have for their denominators one of the numbers 10, 100, 1000, 10,000, &c. If we want to add, for example, $\frac{2}{10}$ and $\frac{43}{100}$ we can say at once $\frac{2}{10}$ is equal to $\frac{20}{100}$, $\frac{20}{100}$ and $\frac{43}{100}$ are equal to $\frac{63}{100}$; if we want to subtract $\frac{2}{10}$ from $\frac{43}{100}$ we take 20 from 43 and find the result $\frac{23}{100}$; if we want to multiply $\frac{2}{10}$ and $\frac{43}{100}$ we have only to multiply 2 and 43, for we know immediately the product of 10 and 100 is 1000, so we easily find the result $\frac{86}{1000}$; lastly, if we want to divide $\frac{43}{100}$ by $\frac{2}{10}$ we reduce the fractions again to a common denominator and find $\frac{43}{20}$. In all cases then the operation is very easy: in multiplication because we can multiply immediately the denominators 10, 100, 1000, &c. together; in addition, subtraction, and division because the fractions can be reduced at once to the same denominator.

Fractions which have one of the numbers 10, 100, 1000, &c. for denominator are called Decimal Fractions.

Since all arithmetical operations are performed more easily on decimal fractions than on other fractions, it is usual to express quantities in terms of their units of measure by means of decimal fractions. This can always be done as accurately as is required for the purpose of the question in hand. For instance, suppose we are dealing with length, and that the actual purpose does not require us to take into account quantity less than $\frac{1}{100}$th of an inch. Every length may be expressed as a certain number of inches and a certain number of hundredths of an inch, together with a small length less than a hundredth of an inch, and this last portion can according to our supposition be neglected.

Hence, as far as our needs in the present case are concerned, we can express every length as so many inches and so many hundredths of an inch. In other cases we may only be able to neglect quantities less than one-millionth of an inch, and then we should express every length as so many inches and so many millionths of an inch. But in all cases the length can be expressed in terms of inches for practical purposes by means of a whole number and a decimal fraction.

Since decimal fractions are used so frequently a special notation is adopted for them. It is merely an extension of that employed for whole numbers. In the number 444 each 4 is equivalent to the tenth part of the 4 which preceded it. Now take the expression 444·44, where the dot is used to separate the whole part from the fractional, and interpret it on the same principle. The 4 coming immediately after the dot must be the tenth part of the 4 in the unit's place, and therefore means four-tenths. The next 4 must be the tenth part of the 4 after the dot, that is the tenth part of four-tenths or four-hundredths. Hence the whole expression will be read four hundred and forty-four, four tenths and four hundredths. The rule then is:

Read the first figure after the dot (this dot is called the decimal point) as so many tenths, the second as so many hundredths, the third as so many thousandths, &c.

Thus ·357 will mean three tenths, five hundredths, and seven thousandths; or with the ordinary notation for fractions $\frac{3}{10}$ and $\frac{5}{100}$ and $\frac{7}{1000}$.

Since $\frac{3}{10}$ is equal to $\frac{300}{1000}$ and $\frac{5}{100}$ to $\frac{50}{1000}$, we may equally well express ·357 as $\frac{357}{1000}$, or three hundred and fifty-seven thousandths.

The expression ·057 will mean: no tenths, five hundredths, and seven thousandths, or simply five hundredths and seven thousandths, which is equivalent to $\frac{57}{1000}$.

·007 will mean no tenths, no hundredths, and seven thousandths, or simply $\frac{7}{1000}$.

·307 will mean $\frac{3}{10}$ and $\frac{7}{1000}$ or $\frac{307}{1000}$.

Obviously ·6 means the same thing as ·60 or ·600, but ·6, ·06, and ·006 mean respectively $\frac{6}{10}$, $\frac{6}{100}$, and $\frac{6}{1000}$.

We shall now explain, first, the mode of converting ordinary fractions into decimal fractions and decimal fractions into ordinary fractions; next, the mode of performing the operations of Addition, Subtraction, Multiplication and Division on decimal fractions.

SECTION II.

CONVERSION OF VULGAR FRACTIONS INTO DECIMAL FRACTIONS.

We will take $\frac{1}{8}$ as an example, and we will suppose for the sake of clearness that we are dealing with French measures, and that we wish to express $\frac{1}{8}$ of a metre in terms of decimetres, centimetres, and millimetres. A decimetre is $\frac{1}{10}$th of a metre, a centimetre is $\frac{1}{10}$th of a decimetre or $\frac{1}{100}$th of a metre, and a millimetre is $\frac{1}{10}$th of a centimetre or $\frac{1}{1000}$th of a metre. Then $\frac{1}{8}$th of a metre is $\frac{1}{8}$th of 10 decimetres or 1 decimetre and $\frac{1}{8}$th of 2 decimetres. 2 decimetres are 20 centimetres, and $\frac{1}{8}$th of 20 centimetres is 2 centimetres and $\frac{1}{8}$th of 4 centimetres. 4 centimetres are 40 millimetres, and $\frac{1}{8}$th of 40 millimetres is 5 millimetres. Hence $\frac{1}{8}$th of a metre is 1 decimetre 2 centimetres and 5 millimetres. Now if a metre be our unit, ·125 means $\frac{1}{10}$th, $\frac{2}{100}$ths, and $\frac{5}{1000}$ths of a metre, or

1 decimetre 2 centimetres and 5 millimetres, so that $\frac{1}{8}$ and ·125 are equivalent expressions. We might have said at once one metre is a thousand millimetres, therefore one-eighth of a metre is 125 millimetres, but the process would have been the same.

Again, take the fraction $\frac{5}{16}$. 5 units are equal to 50 tenths, and the $\frac{1}{16}$th of 50 tenths is 3 tenths, together with the 16th part of 2 tenths or 20 hundredths. The 16th part of 20 hundredths is 1 hundredth, together with the 16th part of 4 hundredths or 40 thousandths. The 16th part of 40 thousandths is 2 thousandths, together with the 16th part of 8 thousandths or 80 ten thousandths. The 16th part of 80 ten thousandths is 5 ten thousandths. Hence $\frac{5}{16}$ is equal to the sum of the fractions $\frac{3}{10}$, $\frac{1}{100}$, $\frac{2}{1000}$, $\frac{5}{10000}$, or in decimal notation $\frac{5}{16}$ is equal to ·3125.

Such an example as the above may be worked conveniently thus:

```
16) 50 (·3125
    48
    ——
     20
     16
     ——
      40
      32
      ——
       80
       80
       ——
```

Here, just as in working division for whole numbers we may leave out all explicit mention of the tenths, hundredths, &c. and say simply 16 into 50 goes 3, 16 into 20 goes 1, and so on.

As another example we take the fraction $\frac{1076}{49}$. We work it thus:

```
49) 1076 (21·95918
     98
      96
      49
      470
      441
       290
       245
        450
        441
          90
          49
          410
          392
           18
```

Here we find by ordinary division that 1076 divided by 49 gives 21 as quotient and 47 as remainder. We replace the 47 by 470 tenths and dividing by 49 obtain 9 tenths. We therefore put a decimal point after the 21 in the quotient and write down 9. Similarly we replace the 29 tenths by 290 hundredths and so on. The last remainder 18 will mean $\frac{18}{100.000}$, and on dividing by 49 the result will be less than $\frac{1}{100.000}$. If we suppose that for the purpose in hand we can neglect quantities less than one hundred-thousandth, we may put $\frac{1076}{49}$ equal to 21·95918; if however greater accuracy is required, it may be obtained by continuing the work farther. But however far we continue the work we shall never obtain a perfectly accurate result; there will always be some remainder. For since 49 is prime to 10 it is also prime to the product of 10 and 10 or to 100, and since it is prime to 10 and to 100 it is prime to the product of 10 and 100 or to 1000; and in the same way it is prime to 10,000, 100,000, 1000,000, &c. Therefore since 49 does not divide 1076 it cannot divide one hundred, one thousand, or one million, &c. times 1076, and $\frac{1076}{49}$ can never be expressed exactly as a decimal fraction.

The only fractions which can be expressed exactly as decimal fractions are those which when reduced to their lowest terms have for their denominator either 2 or 5, or a product formed by multiplying together one or more of these numbers. For when the fraction is reduced to its lowest term the denominator is prime to the numerator, and therefore (if the fraction is to be expressed exactly as a decimal fraction) it must divide one of the numbers, 10, 100, 1000, 10,000, &c. But this is impossible if the denominator has among its prime factor any other numbers besides 2 and 5, such numbers as 3 or 7 for instance.

In other cases after a time the figures in the decimal fractions will repeat themselves over and over again. In the above case, where we divided by 49, our remainders were successively after we came to the decimal point, 29, 45, 41, 18. Now if we had proceeded with the work since all the remainders are less than 49, and there are only 48 numbers (1, 2, 3, 4......48) less than 49, we must at the very latest, after the 48th remainder, have come to one of the old remainders again. But as soon as we came to an old remainder all the quotients and remainders would have repeated themselves. Supposing, for example, the remainder 45 in the fifth line had appeared once more, we should again have had 9 as quotient and 9 as remainder and then 1 as quotient and 41 as remainder, next 8 as quotient and 18 as remainder, and so on.

We may illustrate this by the fraction $\frac{1}{7}$. Here we cannot have more than 6 distinct remainders at the most, and therefore the fraction must at the 7th figure, if not sooner, repeat the old quotients and remainders.

The work will stand thus:

```
7) 10 (·142857
    7
    30
    28
     20
     14
      60
      56
       40
       35
        50
        49
         1
```

Here the remainders are 3, 2, 6, 4, 5, 1, that is to say 6 in all, the most that there could possibly be. But in other cases there will be fewer remainders. Take the vulgar fraction $\frac{1}{3}$ and express it as a decimal fraction.

```
3) 10 (33
    9
    10
     9
     1
```

Here there is only one remainder, namely 1, and we shall obtain for $\frac{1}{3}$ the expression ·3333..., where we continue writing 3's till our approximation is sufficient for the purpose in hand. If we put this result in concrete language we may say, $\frac{1}{3}$rd of a metre is equal to 3 decimetres and $\frac{1}{3}$rd of a decimetre, or to 3 decimetres 3 centimetres and $\frac{1}{3}$rd of a centimetre, or to 3 decimetres 3 centimetres 3 millimetres and $\frac{1}{3}$rd of a millimetre; hence according as we can neglect $\frac{1}{3}$rd of a decimetre, $\frac{1}{3}$rd of a centimetre, or $\frac{1}{3}$rd of a millimetre, we may replace $\frac{1}{3}$ by ·3, by ·33 or by ·333. We may express the same result in ordinary fractional notation. Since 1 is equal to ten-tenths $\frac{1}{3}$ is equal to $\frac{3}{10}$ and $\frac{1}{30}$, or to $\frac{3}{10}$, $\frac{3}{100}$, and to $\frac{1}{300}$, or, by carrying on the same reasoning to $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, $\frac{3}{10000}$, $\frac{3}{100000}$, $\frac{3}{1000000}$, and $\frac{1}{3000000}$; so that $\frac{1}{3}$ only differs from ·333333 by the three millionth part of unity. We see then that though however many 3's we write there will always be a difference between ·333... and $\frac{1}{3}$, yet that difference will continually diminish, and can be made as small as ever we please.

Such an expression as ·33... is called a recurring decimal, and is often denoted by $\cdot\dot{3}$. Similarly $\cdot\dot{2}\dot{7}$ means the fraction that is continually approached though never actually reached by the expression ·27272727..., where we are to continue writing 27's. If more than two figures recur we put a dot on the first and last thus, $\cdot\dot{2}3\dot{5}$ means ·235235235..., and $\cdot 64\dot{7}82\dot{1}$ means ·64782178217821....

There are some simple cases of recurring decimals that it is well to notice.

Take the fraction $\frac{1}{9}$.

$$
\begin{array}{r}
9)\ 10\ (\cdot 11 \\
\underline{9} \\
10 \\
\underline{9} \\
1
\end{array}
$$

Hence $\frac{1}{9}$ is equal $\cdot 11111\ldots$, or to $\cdot\dot{1}$.

Take $\frac{1}{99}$.

$$
\begin{array}{r}
99)\ 100\ (\cdot 0101 \\
\underline{99} \\
100 \\
\underline{99} \\
1
\end{array}
$$

Here 99 will not divide 10, so we put a 0 after the decimal point, signifying that $\frac{1}{99}$ contains no tenths just as the ninety-ninths of a metre contain no decimetre.

We next divide 100 by 99 and find 1 for quotient and 1 for remainder, so that $\frac{1}{99}$ is equal to $\frac{1}{100}$, together with the 99th part of $\frac{1}{100}$. The 99th part of $\frac{1}{100}$ contains no thousandths, but is equal to $\frac{1}{10000}$, together with the 99th part of $\frac{1}{10000}$, and so on.

Thus, $\frac{1}{99}$ is equal to $\cdot 01010101\ldots$, or to $\cdot\dot{0}\dot{1}$.

Again, take $\frac{1}{999}$.

$$
\begin{array}{r}
999)\ 1000\ (\cdot 001\dot{0}01 \\
\underline{999} \\
1000 \\
\underline{999} \\
1
\end{array}
$$

We find $\frac{1}{999}$ equal to $\cdot 001001001\ldots$, or to $\cdot\dot{0}0\dot{1}$.

In the same way $\frac{1}{9999}$ is equal to $\cdot\dot{0}00\dot{1}$, $\frac{1}{99999}$ to $\cdot\dot{0}000\dot{1}$, &c.

Since $\frac{1}{9}$ is equal to $\cdot 11111\ldots$, $\frac{2}{9}$ is equal to $\cdot 222\ldots$ $\frac{3}{9}$ to $\cdot 333\ldots$, $\frac{8}{9}$ to $\cdot 888\ldots$.

Since $\frac{1}{99}$ is equal to $\cdot 010101\ldots$, $\frac{2}{99}$ is equal to $\cdot 020202\ldots$, $\frac{5}{99}$ to $\cdot 050505\ldots$, $\frac{27}{99}$ to $\cdot 272727\ldots$, $\frac{35}{99}$ to $\cdot 353535\ldots$; since $\frac{1}{999}$ is equal to $\cdot 001001\ldots$, $\frac{2}{999}$ is equal to $\cdot 002002\ldots$, $\frac{5}{999}$

to ·005005..., $\frac{27}{999}$ to ·027027..., $\frac{365}{999}$ to ·365365..., and so on.

We may if we please verify these results directly. Thus, in the last case

$$\begin{array}{r} 999)\,3650\,(\cdot 365 \\ \underline{2997} \\ 6530 \\ \underline{5994} \\ 5360 \\ \underline{4995} \\ 365 \end{array}$$

The work shews that $\frac{365}{999}$ is equal to ·365, together with $\frac{365}{999000}$, that is to say, $\frac{365}{999}$ differs from ·365 by less than $\frac{1}{1000}$. By proceeding further we should see that $\frac{365}{999}$ differs from ·365365 by less than $\frac{1}{1000000}$ (one millionth), and by writing more figures in the decimal fraction it can be made to differ from $\frac{365}{999}$ by as small a quantity as we please, so that $\frac{365}{999}$ is equal to $\cdot\dot{3}6\dot{5}$.

Having now explained the mode of converting vulgar fractions into decimal fractions, we proceed to the reverse operation.

SECTION III.

CONVERSION OF DECIMAL FRACTIONS INTO VULGAR FRACTIONS.

This is very easy when the decimal fraction does not recur. We know that ·274 means the same thing as the sum of $\frac{2}{10}$, $\frac{7}{100}$, and $\frac{4}{1000}$ or $\frac{274}{1000}$. We may if we please reduce $\frac{274}{1000}$ to its lowest terms, and we shall find it equal to $\frac{137}{500}$.

Again, $\cdot034$ is equal to $\frac{3}{100}$ and $\frac{4}{1000}$, or to $\frac{34}{1000}$. This may be reduced to $\frac{17}{500}$.

The rule then is,

Take for numerator of the vulgar fraction, the decimal fraction itself considered as a whole number, leaving out any o's which occur before the other figures, and take for denominator of the vulgar fraction 1 with as many o's after it as there are figures in the decimal fraction.

Next, suppose the decimal fraction is a recurring one. We have seen that $\cdot\dot{5}$ or $\cdot555\ldots$ is equal to $\frac{5}{9}$, $\cdot\dot{2}\dot{7}$ or $\cdot2727\ldots$ to $\frac{27}{99}$, and $\dot{3}6\dot{5}$ to $\frac{365}{999}$. The rule then is,

Take for numerator of the vulgar fraction, the decimal fraction itself considered as a whole number, leaving out any o's which occur before the other figures, and take for denominator of the vulgar fraction as many 9's as there are figures in the decimal fraction.

Lastly, we will take the case when only a part of the fraction recurs. The fraction $\cdot1\dot{4}$, or $\cdot1444\ldots$ is an example. This must be equal to the sum of $\frac{1}{10}$, $\frac{4}{100}$, $\frac{4}{1000}$, $\frac{4}{10000}$, &c. Now we know that $\cdot\dot{4}$ or the sum of $\frac{4}{10}$, $\frac{4}{100}\ldots$, is equal to $\frac{4}{9}$; hence $\cdot0\dot{4}$, or the sum of $\frac{4}{100}$, $\frac{4}{1000}\ldots$, &c. must be ten times as small and equal to $\frac{4}{90}$, therefore $\cdot1\dot{4}$ is equal to the sum of $\frac{1}{10}$ and $\frac{4}{90}$. Adding these fractions we obtain the result $\frac{13}{90}$. The numerator of this fraction is the sum of 9 and 4, which differs by 1 from 14, the sum of 10 and 4. Similarly $\cdot1\dot{6}$ is equal to the sum of $\frac{1}{10}$ and $\frac{6}{90}$, or to $\frac{15}{90}$. The numerator as before differs from 16 by 1, and the denominator is again 90. In the same way we shall find $\cdot1\dot{5}$ equal to $\frac{14}{90}$, $\cdot1\dot{2}$ to $\frac{11}{90}$, $\cdot1\dot{7}$ to $\frac{16}{90}$.

Now take a case such as $\cdot3\dot{7}$ where the non-recurring figure is not 1. We shall have as before $\cdot 3\dot{7}$ equal to the sum of $\frac{3}{10}$ and $\frac{7}{90}$. Adding these fractions, the denominator of the resulting fraction will be 90 and the numerator will be the sum of 3 times 9 and 7. But since 9 differs from 10 by 1, 3 times 9 will differ from 3 times 10 or 30 by 3, and the sum of 3 times 9 and 7 will be 3 less than the sum of 3 times 10 and 7, that is to say, 3 less than 37. The numerator of the resulting fraction is therefore 34, and we have $\cdot 3\dot{7}$ equal to $\frac{34}{90}$.

If then we wish to find the value of $\cdot 5\dot{8}$ we take 5 from 58 to find the numerator, and the denominator is 90, so $\cdot 5\dot{8}$ is equal to $\frac{53}{90}$. Similarly we find $\cdot 7\dot{2}$ equal to $\frac{65}{90}$, $\cdot 6\dot{4}$ is equal to $\frac{48}{90}$.

Let us now take a more general case, such as $\cdot 23\dot{6}5\dot{8}$. Here $\cdot 23$ is equal to $\frac{23}{100}$, and $\cdot 00\dot{6}5\dot{8}$ is equal to the hundredth part of $\cdot\dot{6}5\dot{8}$, that is to $\frac{658}{99900}$. So that $\cdot 23\dot{6}5\dot{8}$ is equal to the sum of $\frac{23}{100}$ and $\frac{658}{99900}$. The resulting fraction will have for its denominator 99900, and for its numerator the sum of 23 times 999 and 658. But 23 times 999 will be 23 less than 23 times 1000, and the sum of 23 times 999 and 658 will be 23 less than the sum of 23 times 1000 and 658, or than 23658. Hence $\cdot 23\dot{6}5\dot{8}$ is equal to $\frac{23635}{99900}$.

We have then the following rule for converting a decimal fraction with a recurring part into a vulgar fraction.

To find the numerator of the vulgar fraction, write the decimal fraction as if it were a whole number and take from it the non-recurring part, also written as if it were a whole number; to find the denominator of the vulgar fraction write as many 9's as there are figures in the recurring part, and after them as many 0's as there are figures in the non-recurring part.

Thus we wish to find the value of $\cdot 729\dot{4}\dot{6}$. We first of all take 729 from 72946:

$$\begin{array}{r} 72946 \\ 729 \\ \hline 72217, \end{array}$$

and so find that the numerator is 72217. Next, since there are two figures in the recurring part we write down two 9's, and since there are three figures in the non-recurring part we write after them three 0's and find that the denominator is 99000. Hence $\cdot 729\dot{4}\dot{6}$ is equal to $\frac{72217}{99000}$.

The results found in this way can often be simplified. Thus $23\dot{6}5\dot{8}$ was we saw equal to $\frac{23635}{99900}$, and this may be reduced to $\frac{4727}{19980}$: $1\dot{6}$ is equal to $\frac{15}{90}$ or to $\frac{1}{6}$.

If we applied the rule to the fraction $\cdot 1\dot{1}$ we should obtain $\frac{10}{90}$ or $\frac{1}{9}$. And that this result is correct we see from the fact that $\cdot 1\dot{1}$ means the same thing as $\cdot 11111\ldots$, or as $\cdot\dot{1}$.

Again, the rule will give $\frac{9}{9}$ or 1 for the value of $\cdot\dot{9}$. We may easily prove this to be true directly. For $\cdot\dot{9}$ means the number we continually approach without ever reaching, by taking more and more of the fractions $\frac{9}{10}$, $\frac{9}{100}$, $\frac{9}{1000}$, &c. Now $\frac{9}{10}$ differs from 1 by $\frac{1}{10}$, $\frac{99}{100}$ differs from 1 by $\frac{1}{100}$, $\frac{999}{1000}$ differs from 1 by $\frac{1}{1000}$, and so on.

So that the decimal fraction ·9, ·99, ·999, ·9999, &c. can be made to approach as near 1 as ever we please, but without actually reaching it. And therefore according to definition we put ·$\dot{9}$ equal to 1.

Now we have shewn how to convert vulgar fractions into decimal fractions, and conversely decimal fractions into vulgar fractions, we may explain the addition, subtraction, multiplication and division of decimal fractions.

SECTION IV.

Addition and Subtraction of Decimal Fractions.

To add or subtract decimal fractions we write the figures of the same kind under one another, that is to say tenths under tenths, hundredths under hundredths &c., and proceed in exactly the same way as for whole numbers.

For example we add 24·7296 and 8·6073 thus

$$\begin{array}{r} 24{\cdot}7296 \\ \underline{8{\cdot}6073} \\ 33{\cdot}3369 \end{array}$$

The reason is easily seen. Three ten-thousandths and six ten-thousandths are equal to nine ten-thousandths; seven thousandths and nine thousandths are equal to sixteen thousandths, that is to say, to one hundredth and six thousandths; one hundredth and two hundredths are equal to three hundredths; six tenths and seven tenths are equal to thirteen tenths, that is to say, to one and three tenths; nine and twenty-four are equal to thirty-three. Therefore the sum of the two numbers is equal to thirty-three, three tenths, three hundredths, six thousandths and nine ten-thousandths.

In the same way we should subtract 8·6073 from 24·7296.

24·7296
8·6073
16·1223

The following are examples that the reader can justify by putting into words.

1st, Addition:

5·684237	64·113998
2·052913	7·885423
3·170654	8·567432
10·907804	21·030516
	101·597369

2nd, Subtraction:

3·017285	·654832
·962834	·090988
2·054451	·563844

SECTION V.

Multiplication of Decimal Fractions.

The simplest case is when we have to multiply a decimal fraction by a whole number, for example 3·4892 by 2. This is merely a particular case of addition of decimal fractions. We shall have twice 2 ten-thousandths equal to 4 ten-thousandths, twice 9 thousandths equal to

18 thousandths or to 8 thousandths and 1 hundredth; twice 8 hundredths equal to 16 hundredths, which with the 1 hundredth make 17 hundredths or 7 hundredths and 1 tenth; twice 4 tenths equal to 8 tenths, which with the 1 tenth make 9 tenths; twice three equal to 6. The result is then 6·9784, and the result may stand thus

$$\begin{array}{l} 3{\cdot}4892 \\ 2 \\ \hline 6{\cdot}9784 \end{array}$$

We multiply in the same way as for whole numbers each figure in succession beginning from the right hand side, and we place the figures of the product underneath the corresponding figures of the multiplied number.

Next we will multiply the same number 3·4892 by ·7. To multiply by ·7 or $\frac{7}{10}$ is the same thing as dividing by 10 and multiplying by 7. Now to divide 3·4892 by 10 we need only move each figure one place further to the right and the result will be ·34892. We now multiply by 7 as before.

$$\begin{array}{l} {\cdot}34892 \\ 7{\cdot} \\ \hline 2{\cdot}44244 \end{array}$$

Again, let us multiply 3·4892 by ·06 or $\frac{6}{100}$. We move each figure two places to the right and multiply by 6.

$$\begin{array}{l} {\cdot}034892 \\ 6{\cdot} \\ \hline {\cdot}209352 \end{array}$$

To multiply 3·4892 by ·001 or $\frac{1}{1000}$ we need only move each figure three places to the right, and we thus obtain ·0034892.

Lastly, to multiply 3·4892 by ·0005 or $\frac{5}{10000}$ we move each figure four places to the right and multiply by 5.

$$\begin{array}{r} \cdot00034892 \\ \underline{5\cdot} \\ \cdot00174460 \end{array}$$

We should proceed in the same way for multiplication by single figures in other places.

If now we wish to multiply 3·4892 by 2·7615 we have only to add the preceding results,

$$\begin{array}{l} 6\cdot9784 \\ 2\cdot44244 \\ \cdot209352 \\ \cdot0034892 \\ \underline{\cdot00174460} \\ 9\cdot63542580 \end{array}$$

But the whole work may be written more compactly thus,

$$\begin{array}{l} 3\cdot4892 \\ \underline{2\cdot7615} \\ 6\cdot9784 \\ 2\cdot44244 \\ \cdot209352 \\ 34892 \\ \underline{174460} \\ 9\cdot63542580 \end{array}$$

We write the figures of the multiplied number and multiplier one under the other, units under units, tenths under tenths, &c. We then multiply by 2, placing the figures of the product under the corresponding figures of the multiplicand; then we multiply by 7, placing the figures of the product one place further to the right than those of the multiplicand; then by 6, placing the figures of the product still another place further to the right, and so on.

The 0 at the end of the result has of course no meaning and may be left out.

As another example we will multiply ·75 by ·56,

```
 ·75
 ·56
 ·375
   450
 ·4200
```

In this case there is no figure in the units' place of the multiplier, so we multiply by 5 and put down the figures of the product one place further to the right than the corresponding figures of the multiplicand. The final result is ·42, since the 0's which occur have no meaning.

Take next the product of ·43 and ·087,

```
 ·43
 ·087
   344
    301
 ·03741
```

Here there are no figures either in the units' place or in the first decimal place, so we multiply by 8 and put the figures in the result two places further to the right than the corresponding figures of the multiplicand. Thus 8 times 3 are 24, and instead of putting the 4 under the 3 we put it two places further to the right. When we multiply by 7 in the next lines we put the figures of the product one place further still to the right. Since there are no figures of the product in the first decimal place we must put a 0 there, for the absence of any other figure means that there are no tenths and this is expressed by writing a 0 in the result. The 344 in the first line of course means ·0344 and the 301 in the second ·00301, but for brevity we leave out the 0's in the intermediate steps and simply write them in the result when necessary.

We will now take a case where hundreds and tens occur in the numbers to be multiplied. We will multiply 78·92 by 324·07 for example.

Here we may multiply by the whole number 324 in the ordinary way, beginning with the 4, then multiplying by the 2 and moving each figure one place to the left, then by the 3 moving each figure another place to the left. But it is best to perform the operation in the reverse way, beginning with the 3 and ending with the 4, as follows:

$$\begin{array}{l}
78{\cdot}92 \\
\underline{324{\cdot}07} \\
23676 \\
1578{\cdot}4 \\
315{\cdot}68 \\
\underline{5{\cdot}5244} \\
25575{\cdot}6044
\end{array}$$

In each line we move a place further to the right and the work is thus continuous with the multiplication by the decimal part which follows it. Moreover the most important parts of the product are formed first, and so the process is a continual approximation. Of course when we multiply by the 3, since this 3 means 300 we place the figures of the product two places further to the left than the corresponding figures of the multiplicand. We place the 6 under the 8 of 78·92, not under the 2. It must be noticed in all these examples that the decimal points of the multiplicand, multiplier, and product are in the same vertical line.

In almost all practical cases where decimal fractions are employed, we need only approximate results. The numbers we start with, if they arise from the measurement of actual quantities, cannot be absolutely accurate. If for example the number 3·4892 were given us as the length of a line in feet, it would be only meant that the line differs from 3·4892 feet by less than $\frac{1}{10000}$ of a foot. What the actual length in feet is we do not know, it might be 3·489247 or 3·489238 or any other number whose whole part and first four decimals are 3·4892. When now in practice we wish to multiply decimal fractions such as 3·4892 and 2·7615, it is useless working out the result to eight decimal places, since only four of the figures after the decimal point can be correct. We therefore proceed as follows:

$$
\begin{array}{r}
3{\cdot}4892 \\
\underline{2{\cdot}7615} \\
6{\cdot}9784 \\
2{\cdot}4424 \\
{\cdot}2093 \\
35 \\
\underline{17} \\
9{\cdot}0353
\end{array}
$$

We first multiply by 2 as usual, we then multiply by 7 beginning with the 9 and only taking the last figure 2 into account so far as it furnishes us with something to carry, we next multiply by 6 beginning with the 8, then with 1 beginning with the 4, and lastly with 5 beginning with 1. In the second line we say 7 times 9 are 63 and 1 (carried from 7 times 2 are 14) are 64; in the third we say 6 times 8 are 48 and 5 (carried from 6 times 9 are 54) are 53. In the fourth line the reader will notice that instead of putting 34 we put 35. This is because 3·48 is nearer 3·5 than 3·4. Moreover, if we always put the next lowest number our errors would all be errors of defect and would mount up. But by the present method the errors are sometimes errors of defect and sometimes errors of excess, and therefore in part neutralize one another. Our former statement about the meaning of 3·4892 requires then a slight correction. When we are working to four decimal places we put 3·4892 for any number which differs from that decimal fraction by less than $\frac{1}{20000}$ either in excess or defect, not merely as was said before by less than $\frac{1}{10000}$. The number for which we put 3·4892 may have any value between 3·48915 and 3·48924. If we stopped at the 5, there would be no reason for putting 3·4892 rather than 3·4891 in the place of 3·48915, but in practice there will always be figures following the 5 and these will make the number nearer 3·4892.

By comparing the result 9·6353 with the result 9·6354458 we obtained by multiplying the numbers in full, it will be seen that the last figure 3 in the approximate calculation is not correct. This will often be the case. The errors we make, though each separately is less than one half a unit in the last place, will often add up so as to become more than a unit or even than two or three units in the last place. Hence the rule is always to work to one more place of decimals than the number of places we require to be correct. In the present instance, as we are working to four places of decimals, we only expect three places to be correct. And apart from errors of calculation the errors in the numbers we start with may make our last figure wrong. To take a simple example; we will multiply ·9 by ·8, only working to one place of decimal. Since ·9 multiplied by ·8 is ·72 we should put for the result ·7. But ·9 means any number between ·85 and ·94, and ·8 means any number between ·75 and ·84. Let us take the extreme values and multiply them at full length.

·85	·94
·75	·84
·595	·752
425	376
·6375	·7896

The results would be expressed to one place of decimals by ·6 and ·8 and would each differ from ·7 by a unit. So that if our initial numbers are only given to the first figure of decimals we cannot be sure of the first figure of decimal in the results.

The same principle will apply to the multiplication of whole numbers, when these numbers are not obtained from counting distinct objects but from measuring distances, areas, &c., and are therefore only approximately correct. For instance, suppose we wish to multiply 856294 by 3486. Here 3486 may be either too great by ·5 or $\frac{1}{2}$ or too small by $\frac{1}{2}$, and these errors would on multiplication produce an error in the result of 428147 either in excess or defect. In this case then only the first four figures of the result will be correct and our multiplication should be effected so as to obtain only these.

We perform the operation thus,

```
    856294
      3486
  --------
  25689
   3425
    685
     51
  -----
  29850
```

In this case multiplication at full length would give 2985040884, so that as far as the arithmetic is concerned all the figures of the result are correct. But we only retain four of these since, as we have already seen, the possible error in the original numbers will affect the fifth figure. We say then that the product is 2985 millions and expect an error of some few hundred thousands. Cases actually arise where the quantity to be measured is so great compared with the unit of measurement. The distance of the sun from the earth for instance is said to be 92 millions of miles, but it may be some hundred thousands of miles more or less.

If in the above example we had only worked for four figures, the result would have been 2986, and the fourth figure would have been wrong. This shews the necessity of always working for one more figure than we require.

As a last example we will multiply to eight places of decimals (and therefore correct to seven) ·43429448 by 2·30258509,

$$\begin{array}{r}
\underline{\cdot 43429448} \\
2\cdot 30258509 \\
\cdot 86858896 \\
\cdot 13028834 \\
86859 \\
21715 \\
3474 \\
217 \\
\underline{4} \\
\cdot 99999999
\end{array}$$

The result is therefore 1 as far as the seventh place of decimals.

SECTION VI.

Division of Decimal Fractions.

In considering the division of one decimal fraction by another only two new questions arise. The first relates to the position of the decimal point in the quotient. The following rule will give this:

Move the decimal point in both dividend and divisor the same number of places to the right or left, till in the divisor it comes immediately after the figure in the units' place, then the decimal point in the quotient will correspond to the decimal point in the dividend.

For instance, we wish to divide ·0024 by ·064. We move the decimal point two places to the right in both numbers and make them ·24 and 6·4. We have merely multiplied both numbers by 100, so that their quotient will be unaltered. The work will now be as follows:

```
6·4) ·240 (·0375
     192
      480
      448
       320
       320
```

We put a decimal point in the quotient's place and see if 6·4 can be taken from 2. As it cannot we put a 0 in the quotient and inquire how many times 6·4 can be taken from 24. The answer is three times, and we put 3 in the quotient. It is clear in fact that if when we divide 24 by 6·4 we obtain a quotient 3, we shall obtain a quotient ·03 or $\frac{3}{100}$ when we divide ·24 or $\frac{24}{100}$ by 6·4. The work proceeds as in division of whole numbers.

The second question relates to the method of performing division when we only expect the result to be accurate to a certain number of decimal places.

We will illustrate this by dividing ·06314 by ·007241 to six places of decimals,

```
7·241) 63·140 (8·719790
       57·928
        5·2120
        5·0687
         14330
          7241
          70890
          65169
           5721
           5069
            652
            650
```

When we have written 70890 (after the third line of division) we have already used six decimal places of the dividend, so we bring down no more o's; but when we multiply by 7, the next figure of the quotient, we begin with the 4 of the divisor instead of with the 1, and when we multiply by the succeeding 9 of the quotient we begin with the 2 of the divisor. The result is correct to five places of decimals.

As another example we will divide to seven places of decimals 1·9487632 by 2·3025851.

```
2·3025851) 1·9487632 (·8463371
           1·8420681
           ---------
             1066951
              921034
             -------
              145917
              138155
              ------
                7762
                6908
                ----
                 854
                 691
                 ---
                 163
                 161
                 ---
                   2
                   2
                   -
```

In multiplying by ·8, the first figure of the quotient, we begin with the last figure but one of the divisor, since the product of the last figure 1 and ·8 will fall beyond the seventh place of decimals. We carry however 1 from this product, since 8 is nearer 10 than 0. Again, when multiplying by the next figure 4 of the quotient we begin the last figure but two of the divisor, carrying however

2 from the 4 times 5. In this way we continue the operation, beginning each time with a figure of the divisor one place further to the left.

The result is correct to the last place.

EXAMPLES.

1. Find to **7** decimal places the product of ·6981543 and ·8284786.

2. Find to **5** decimal places the product of ·78817 and ·52915.

3. Find to **7** decimal places the product of 10·3321808 and ·4342945.

4. Find to **7** decimal places the product of 4·4872091 and 2·3025851.

5. Find to **8** decimal places the product of ·21492764 and ·31731704.

6. Find to **12** decimal places the product of ·894563127357 and ·506830986731.

7. Divide to 6 places of decimals 8·987678 by 7·965341.

8. Divide to 4 places of decimals 77936 by 32827.

9. Divide to 8 places of decimals 1 by 5·3214678.

10. Divide exactly ·035 by ·0007.

11. Divide exactly ·72 by ·04 and by 40.

12. The whole numbers which express most nearly in yards the length and breadth of an area are 4328 and 2976; what is the greatest possible error that can be made in estimating the number of square yards in the area?

13. The sign – put over the last figure in a number has been used sometimes to indicate that the figure is too great, and then the absence of any sign indicates that the figure is too small. With this notation what errors may be made in multiplying (*a*) $4{\cdot}68230\bar{7}$ by $2{\cdot}46182\bar{9}$, (*b*) $4{\cdot}68230\bar{7}$ by 2461829, (*c*) $4{\cdot}682307$ by $2{\cdot}46182\bar{9}$, (*d*) $4{\cdot}682307$ by $2{\cdot}461829$?

CHAPTER VI.

Powers and Roots.

SECTION I.

Powers of Numbers.

We have seen that the addition of equal numbers is called multiplication. We may now consider the results of multiplying equal numbers. These results are called the powers of number. Just as the result of adding four threes together is said to be three multiplied by four, so the result of multiplying four threes together is said to be three raised to the fourth power.

A number is said to be raised to the power indicated by another number when we multiply the first number by itself as many times as there are units in the second number.

If we simply multiply a number by itself we raise it to the second power. This second power is often called the square of the number. Thus 7 times 7 or 49 is the square or second power of 7. The following are the squares of the first ten numbers,

1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Multiplying these squares by the corresponding numbers we obtain the third powers, or cubes as they are generally called, of the first ten numbers. They are

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

Multiplying the cubes by the numbers themselves we obtain the fourth powers of the first ten numbers,

1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10,000.

In the same way we may go on to find the 5th, 6th, 7th powers of the different numbers.

It must not be supposed that raising a number to a power stands to multiplication as multiplication itself does to addition. We can see at once that the commutative law does not hold. The second power of three is nine, whereas the third power of two is eight. The third power of four is sixty-four, whereas the fourth power of three is eighty-one. The reason of this is that we can suppose every number to be formed by the addition of units and therefore multiplication or the addition of the sums of these units may be considered to be a sort of double addition. But we cannot suppose every number to be formed by the multiplication of units, and therefore raising to a power cannot be considered to be a sort of double multiplication. Raising to a power is not the multiplication of the product formed by multiplied units as it should be if the analogy were complete, but it is the multiplication of the sums formed by added units.

We shall soon require the following important theorem relating to the squares of numbers.

The square of the sum of two numbers is equal to the sum of the squares of the number together with twice the product of the numbers.

For instance 11 is the sum of 6 and 5, and the square of 11 will be equal to the square of 6 and the square of 5 together with twice the product of 6 and 5.

Now by the square of 11 we mean 11 multiplied by 11 or the sum of eleven 11's.

We will write these 11's in two vertical lines, six in one line and five in another, thus:

11,	11
11,	11
11,	11
11,	11
11,	11
11,	

Next we write in the place of each 11 the numbers 6 and 5 of which it is the sum. Then the square of 11 will be the sum of the following numbers:

6,	5,	6,	5,
6,	5,	6,	5,
6,	5,	6,	5,
6,	5,	6,	5,
6,	5,	6,	5.
6,	5,		

We have now four vertical columns of numbers to be added together. The first column is equal to 6 times 6 or the square of 6. The last column is equal to 5 times 5 or the square of 5. The second column is equal to 6 times 5 and the third to 5 times 6. These two middle columns are equal and their sum is twice the product of 5 and 6.

All the columns together make then the square of 6, twice the product of 5 and 6, and the square of 5, and this sum is therefore equal to the square of 11.

The ordinary method of multiplication does not admit of simplification when the numbers to be multiplied are equal. There is no rule for finding powers similar to the rule for multiplication. When however we wish to raise a number to the 4th or 5th or some higher power, we may shorten the operation in practise by first forming a table of the multiples of the number. For instance we wish to raise ·47162 to the 5th power, working to five places of decimals. The following will be the work:

1	·47162
2	·94324
3	1·41486
4	1·88648
5	2·35810
6	2·82972
7	3·30134
8	3·77296
9	4·24458

·18865	·09432
3301	943
47	94
28	19
1	1
·22242—(*a*)	·10489—(*b*)
·04716	·01886
189	424
38	19
4	3
·04947—(*c*)	·02332—(*d*).

Here (*a*) is the square, (*b*) the cube, (*c*) the 4th power, (*d*) the 5th power of ·47162. We multiply this number first by itself, then by its square, then by its cube, then by its fourth power. The work is set down in the four sums above just as in ordinary multiplication except that the numbers multiplied are not written at the top of each sum.

The table is formed as follows: We double the 1st line to form the 2nd, add the 1st and 2nd to form the 3rd, double the 2nd to form the 4th, add the 2nd and 3rd to form the 5th, double the 3rd to form the 6th, add the 3rd and 4th to form the 7th, double the 4th to form the 8th, add the 4th and 5th to form the 9th. There are several obvious methods of verification.

The following is a further illustration. Find the results of dividing the square ·2623643 by 2, the cube by 2 and then by 3, the 4th power by the product of 2, 3 and 4, the 5th power by the product of 2, 3, 4, 5 and so on. Continue the process as long as figures within the seventh place of decimals are obtained and add all the results to the number itself.

1	·2623643
2	·5247286
3	·7870929
4	1·0494572
5	1·3118215
6	1·5741858
7	1·8365501
8	2·0989144
9	2·3612787

```
  ·0524729            ·0078709
    157419               10495
      5247                1049
       787                  26
       157                  18
        10                   1
         1            3)·0090298
2)·0688350              ·0030099—(b)
  ·0344175—(a)

  ·0007871            ·0000262
        24                 236
         2                  18
4)·0007897                   1
  ·0001974—(c)        5)·0000517
                        ·0000103—(d)

        26                7)1
         1                  0—(f)
      6)27
         5—(e)

        ·2623643
          344175     (a)
           30099     (b)
            1974     (c)
             103     (d)
               5     (e)
        ·2999999
```

The result ought to be ·3 so there is only an error of 1 in the seventh place.

We square the number and divide by 2 and thus obtain (*a*). We next multiply the number by (*a*), this gives one half of the cube; we divide by 3 and obtain (*b*) the sixth part of the cube. We multiply by (*b*) and divide by 4 to obtain (*c*). By continuing the process (*d*), (*e*), (*f*) are obtained. The last result (*f*) has no figure in the 7th decimal place and so may be neglected with all that follow*.

It will be noticed that in finding (*e*) we have omitted the o's which precede the 26. This might have been done with all the other multiplications, for the meaning of the figures is sufficiently shewn by the last figure being always in the 7th decimal place. It saves some time in practice when always working to a given number of places to leave out both the decimal points and any o's which occur before the significant figures.

SECTION II.

GEOMETRICAL ILLUSTRATIONS.

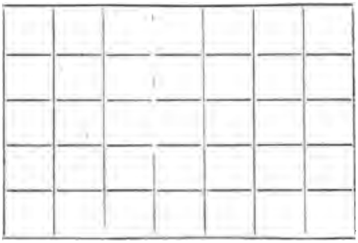
THE terms product, square, cube can be illustrated geometrically. Suppose we have lines whose lengths are expressed in terms of a certain unit of measure by the numbers 5 and 7. We may imagine these lines to form the sides of a lid of a box and we will take an inch as the unit. Then it will be seen from the figure that the lid can

* The reader who is acquainted with the elements of algebra will see that we have found the value of

$$x+\frac{x^2}{2}+\frac{x^3}{2\cdot 3}+\frac{x^4}{2\cdot 3\cdot 4}+\dots$$

where $x = \cdot 2623643$.

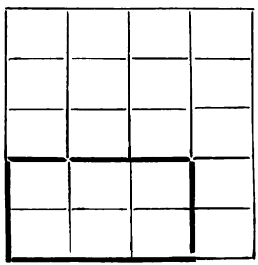
be divided in 35 equal compartments and each of those compartments is a square inch.



The area of the lid is therefore 35 square inches, so that by multiplying together the numbers which express the lengths of the sides we obtain the number which expresses the area in terms of square inches.

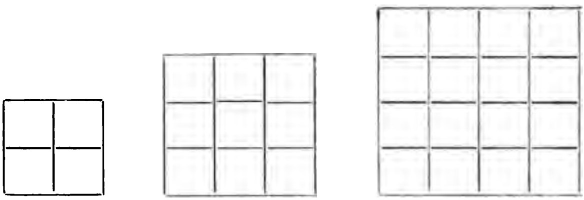
In the same way the surface of an oblong table 4 feet long and 6 feet broad will contain 24 square feet, the area of a field 90 yards long and 80 yards broad will be 7200 square yards, &c.

This meaning of the product of two numbers will apply to the case of fractions. For instance we see from the figure below that the area of a rectangle whose sides are $\frac{2}{5}$ and $\frac{3}{4}$ of an inch is $\frac{6}{20}$ of a square inch.



The whole square represents a square inch. It is divided into 20 equal rectangles each of which is equal therefore to the twentieth part of a square inch. The darker lines in the figure are respectively $\frac{2}{5}$ths and $\frac{3}{4}$ths of an inch and the rectangle contained by them is equal to 6 small rectangles or to $\frac{6}{20}$ths of a square inch. We seem thus to have a new reason for the rule for the multiplication of two fractions, but this reason will also be found to depend ultimately on the associative principle.

A particular case of the product of two numbers is when the numbers are equal. The squares formed on lines whose sides are respectively 2, 3, 4 inches will as the figures shew contain 4, 9, 16 square inches.

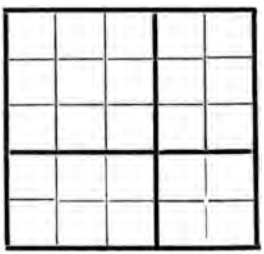


Thus the number of square inches contained in a square is found by taking the second power of the number of inches in the side of the square. It is from this connection with the square in geometry that the second power of a number is called its square in arithmetic.

The product of three numbers can be also illustrated geometrically. Suppose we have a box whose depth is 3 inches and whose length and breadth are respectively 7 and 5 inches. By horizontal partitions we can divide that box into three equal compartments each of which will be an inch in depth. Now we saw that the lid or base of the box can be divided into 5 times 7 or 35 equal squares each of which will be an inch both ways. Hence by drawing vertical partitions we divide each of the 3 compartments into 35 equal cubes and so the whole box will contain 3 times 35 or 3 times 5 times 7 cubic inches.

To find then the number of cubic inches in a box, we must multiply together the lengths of its sides expressed in inches. Thus the number of cubic feet in a room 10 feet high, 15 feet broad and 18 feet high will be 10 times 15 times 18 or 2700. When the length, breadth and depth are equal the solid becomes a cube. The number of cubic inches in a cube whose side is 2 inches will be 2 multiplied by 2 multiplied by 2 or 8; in a cube whose side is 3 inches it will be 3 times 3 times 3 or 27 and so on. To find then the number of cubic inches in a cube we take the third power of the number of inches in its side. This is the reason why the term cube is applied in arithmetic to the third power of a number.

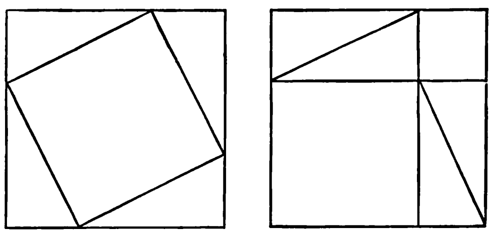
The theorem given in the last section respecting the square of the sum of two numbers is illustrated by the accompanying figure.



Let the numbers be 3 and 2, and their sum 5. We draw a line 5 units in length and construct a square on it. We divide this square as shewn in the figure into four parts, namely the square on a line 3 units in length, the square on a line 2 units in length and 2 oblongs, the sides of each of which are 3 units and 2 units. Now the number of square units in the whole square is the square of 5, the number of square units in the small squares are the square of 3 and the square of 2 respectively, and the number of square units in each of the rectangles is the product of 3 and 2. Hence the square of 5 is equal to the square of 3 together with the square of 2 and twice the product of 3 and 2.

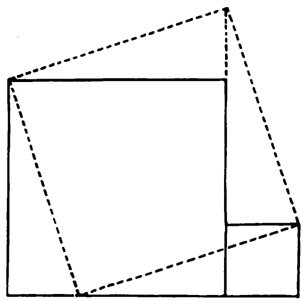
The following geometrical theorem gives rise to important arithmetical questions.

The square on the longest side of a right-angled triangle is equal to the sum of the squares on the other sides.



This may be seen by comparing the two figures above. Each figure represents a square and from that square four triangles are supposed to be cut off, but they are cut off in a different way in the two figures. In the first case we have left a single square and in the second two smaller squares. Since then the large squares in both figures are equal and the triangles cut off are equal, the single square which is left in the first figure must be equal to the sum of the squares which are left in the second figure.

The following figure shews the same theorem in another way.



We take two squares in contact with one another, so as to form one piece as indicated by the unbroken lines. From the whole piece we cut off two triangles and put them on in different places as shewn by the dotted lines. While still using exactly the same material we have thus converted the two squares we started with into a single square.

The reader may easily convince himself of the truth of the theorem by cutting out the squares in pasteboard.

If we know the lengths of two of the sides we can find the third side. To take a simple case known in most countries and discovered probably before the general theorem ;

The two shorter sides of a right-angled triangle are 3 inches and 4 inches, what is the length of the longest side ?

The square on the side 3 inches long will contain 9 square inches, and the square on the side 4 inches long will contain 16 square inches, therefore the square on the longest side must contain 9 and 16 or 25 square inches and the longest side must be 5 inches in length.

We see from this example that if three numbers represent the lengths of the three sides of a right-angled triangle measured in inches, or some other unit of measure, the sum of the *arithmetical* squares of two of these numbers must be equal to the *arithmetical* square of the third. This suggests the question of finding numbers so related that the squares of two of them are together equal to the square of the third. Two rules of very early date have been given for this purpose. The first is attributed to Pythagoras.

Take any odd number for the first number, take one from its square and halve the result for the second number, add one to its square and halve the result for the third number.

For example square the odd number 3 and we have 9; take 1 away and halve, we have 4 ; add 1 and halve, we have 5. And 3, 4, 5 will have the property required. Again the square of 5 is 25, take away 1 and halve, we have 12, add 1 and halve, we have 13. And 5, 12, 13 may be the sides of a right-angled triangle.

The number 7 gives 7, 24, 25 and 9 gives 9, 40, 41.

The second rule is attributed to Plato.

Take any even number for the first number, subtract one from the square of half the number for the second number, add one to the square of half the number for the third number.

Thus take 4; its half is 2; square and subtract 1 we have 3; square and add 1 we have 5. Thus 4, 3, 5 are the required numbers. Again take 8; its half is 4, square and subtract 1 we have 15; square and add 1 we have 17; and it will be found that the square of 8 together with the square of 15 is equal to the square of 17.

The general rule however which includes both the preceding rules is as follows:

*Take any two numbers, form the difference of their squares, twice their product and the sum of their squares; we shall then have the three numbers required.

For example take 5 and 2, the difference of their squares is 21, twice their product is 20 and the sum of their squares is 29. We shall find that 21, 20 and 29 have the required property.

It is obvious that we may multiply all three numbers found by any of the above rules by any common factor

* This last rule, if common factors be left out of account, gives all possible numbers of the kind required. As the reader acquainted with algebra may prove, the most general expressions for such numbers are $s(m^2 - n^2)$, $2smn$, $s(m^2 + n^2)$ where s, m, n are whole numbers.

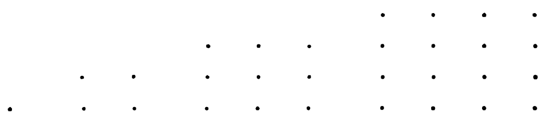
we choose without altering the property they possess of being the sides of a right-angled triangle. Thus we can multiply 3, 4, 5 each by 2, and make them 6, 8, 10. Since the square of each one will be increased 4 times the sum of the squares of two will still be equal to the square of the third.

We may mention here two simple theorems connected respectively with squares and cubes.

We first of all write down the odd numbers and add them together beginning with 1 and not omitting any. At whatever point we stop the sum will always be a square number. We will write the sum below the last odd number added.

1	3	5	7	9	11	13	15
1	4	9	16	25	36	49	64

The sum of 1 and 3 is 4, the sum of 4 and 5 (or 1, 3 and 5) is 9, the sum of 9 and 7 is 16, &c. Since we have nothing to add to 1 we write underneath it another 1 and then we have in the second line the whole series of squares. It will be noticed that when we add two numbers together we have the square of two, three numbers together the square of three, four numbers together the square of four and so on. The following figures will illustrate this property of numbers :



We build up the squares by first putting a dot in the corner and then adding in successive sets dots all round it. Each time we add an odd number of dots.

For the second theorem we write the odd numbers down and then separate them off in groups by vertical lines.

1	3, 5	7, 9, 11	13, 15, 17, 19	21, 23, 25, 27, 29	31, 33, 35, 37, 39, 41
1	8	27	64	125	216

The first number is of course itself the cube of 1, the sum of the next two numbers is the cube of 2, the sum of the three following numbers is the cube of 3, the sum of the four numbers following after them is the cube of 4 and so on.

This theorem is first found in the Arithmetic of Nicomachus of Gerasa written in the first or second century of our era and should be known as the theorem of Nicomachus.

SECTION III.

Square Roots and Incommensurable Quantities.

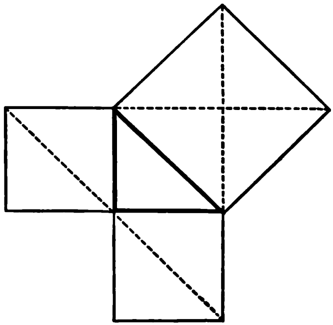
The square root of a given number is a number whose square is equal to the given number.

Thus 2 is the square root of 4, 3 is the square root of 9, 4 of 16, &c.

If we take the series of the whole numbers we see that most of them cannot have a whole number as square root. The square roots of 2 and 3 cannot be whole numbers because the square of 1 is less than 2 or 3 and the square of 2 is greater than 2 or 3; the square roots of 5, 6, 7 cannot be whole numbers for the square of 2 is less than 5, 6 or 7 and the square of 3 is greater than 5, 6 or 7. But these square roots cannot be fractions either, for we can shew that the square of a fraction must always itself be a fraction. Take any fraction and suppose it reduced to its lowest terms so that numerator and denominator are prime to one another, the square of the fraction will be equal to the square of the numerator divided by the square of the denominator. But since the denominator is prime to the numerator it is also prime to the square of the numerator and therefore neither the denominator itself nor its square can divide the square of the numerator exactly. Hence the square of a fraction cannot be a whole number.

Can then any meaning be attached to the expression square root of 2, for example: since this square root can be neither a whole number nor a fraction?—We *can* find a geometrical meaning for the expression square root of 2. Take a right-angled triangle each of whose shorter sides is one inch in length, then the square on the longest side will be equal to 2 square inches. This is a particular case of the general proposition proved in the last section and it may be shewn directly from the following figure.

The dark lines represent a right-angled triangle, with two equal sides. On each of the sides of the triangle a square is constructed and the squares are divided into triangles. The large square contains four of these



triangles and each of the small squares will contain two of these triangles. Therefore the large square is equal to the sum of the small squares and if the small squares be each a square inch, the large square will contain two square inches. Hence the square root of 2 which is commonly written $\sqrt{2}$ may be considered to be the length, expressed in inches of the longest side of a right-angled triangle, when each of the shorter sides is one inch; or in other words $\sqrt{2}$ may be considered to be the length of a diagonal of a square when each of the sides is 1.

Now it is obvious geometrically that according as a line is greater or less than another line, its square is greater or less than the square of that line; and conversely according as the square of a line is greater or less than the square of another line so is the first line greater or less than the second line. A line $\sqrt{2}$ inches in length must then be greater than any line whose square is less than 2 square inches, and less than any line whose square is greater than 2 square inches. For example $\sqrt{2}$ inches is greater than 1 inch since 1 square inch is less than 2 square inches, and less than $\frac{3}{2}$ inch, since $\frac{9}{4}$ square inches are greater than 2 square inches. Since this relation is independent of the special unit chosen whether inch, foot, yard, &c., we may say simply $\sqrt{2}$ is greater than 1 and less than $\frac{3}{2}$. And we have seen that the number of square inches, feet, &c., in a square is the arithmetical square of the number of inches, feet, &c., in the side of the square, whether that number be whole or fractional. We come then to the following arithmetical definition of $\sqrt{2}$.

The square root of 2 is not equal to any number, whole or fractional, but is always greater than any number whose square is less than 2, and less than any number whose square is greater than 2.

Thus like the whole number 6 or the fraction $\frac{2}{3}$ or any other symbol, $\sqrt{2}$ is defined by its fundamental relations to other symbols.

Though $\sqrt{2}$ cannot be expressed exactly by any fraction we can find fractions which will differ from it by less than any number we please.

For instance, we want to express $\sqrt{2}$ approximately as so many fifths. We multiply 2 by 25 and find the square root of the square number nearest to 50. In this case it is 7, and since 2 is greater than the square of $\frac{7}{5}$ or $\frac{49}{25}$, $\sqrt{2}$ is greater than $\frac{7}{5}$. But since 2 is less than $\frac{64}{25}$, $\sqrt{2}$ is less than $\frac{8}{5}$ and therefore $\sqrt{2}$ must differ from $\frac{7}{5}$ by less than $\frac{1}{5}$.

So if wanted to find a fraction differing from $\sqrt{2}$ by less than a tenth, a hundredth, a thousandth, a millionth, &c., we should multiply 2 by the square of ten, one hundred, one thousand, one million, &c. and find the square root of the nearest square number. We can thus find fractions approaching $\sqrt{2}$ as closely as we please.

The general method of carrying out such calculations will be explained in the next section, but a special mode of approximating to $\sqrt{2}$ deserves mention from its early date and from being a simple case of a general theory discovered long afterwards. It is given by Theon of Smyrna, an arithmetician who lived in the beginning of the 2nd century of our era.

We form two rows of numbers each beginning with 1, and write them one above the other. The pair of numbers in each vertical line is formed from the preceding pair thus: we add the two numbers to make the next lower number; we add the upper number to twice the lower to make the next upper number. The rows will be,

1	3	7	17	41	...
1	2	5	12	29	...

since 1 and 1 make 2, 1 and twice 1 make 3, 3 and 2 make 5, 3 and twice 2 make 7, &c. The numbers in the vertical lines are respectively the numerators and denominators of fractions which continually approach $\sqrt{2}$ and which are alternately greater and less than $\sqrt{2}$. Thus $\frac{3}{2}$ is greater than $\sqrt{2}$, $\frac{7}{5}$ is less than $\sqrt{2}$, $\frac{17}{12}$ is greater than $\sqrt{2}$, &c.

Since no whole number can have a square double the square of another whole number, it is impossible that the side and diagonal of a square can be expressed at once as whole numbers in terms of any unit of measure. In other words, the side and diagonal of a square cannot both be measured exactly by the same straight line; they can have no common measure. Two lines so related are called incommensurable. Of course such a relation is only a conception; it cannot be shewn to exist between the lines we meet with in actual experience. Any real physical line can be expressed by some whole number or fraction in terms of any other. For instance, if the hundredth of an inch were the utmost length we could distinguish by the eye we should when measuring with no other help express every line as so many hundredths of an inch. The diagonal of a square whose side was an inch would appear to be $1\frac{41}{100}$ inches in length. We can only create a difference by thought, and we do so by assuming as true beyond the limits of the senses properties which we perceive to be true within those limits.

A line which is incommensurable with the unit of measure is said to contain an incommensurable number of these units. It follows that the essential property and therefore definition of an incommensurable number is that it can be approximated to indefinitely but never expressed exactly by means of fractions. The square roots of 2, 3, 5, are examples of such numbers.

The cube root of a given number is a number whose cube is equal to the given number. Thus 2 is the cube root of 8, 3 is the cube root of 27, 4 of 64, &c.

The fourth root of a given number is a number whose fourth power is equal to the given number, and similar definitions apply to all other roots. Thus 5 is the 4th root of 625, 6 is the 4th root of 1296, 4 is the 5th root of 1024, 3 is the 6th root of 729, 2 is the 7th root of 128, &c.

The cube roots of 2, 3, 4, 5, 6, 7 are clearly not whole numbers, and it may be proved that they cannot be fractions. The cube root of 2 can however be expressed as nearly as we please by fractions if we assume that it is greater than any fraction whose cube is less than 2 and less than any fraction whose cube is greater than 2, and this also applies to the other cube roots. These cube roots are therefore what we have called incommensurable numbers. So likewise are the fourth roots of all the whole numbers from 2 to 15, the fifth roots of all the whole numbers between 2 and 63, &c.

Besides the roots of whole numbers and fractions there are other incommensurable numbers. The number which expresses the circumference of a circle in terms of its diameter is an example. This number is greater than 3 and less than 4, greater than 3·1 and less than 3·2, greater than 3·14 and less than 3·15, and we can continue the approximation as far as we wish. But we can never express the number exactly by means of any fraction.

When the length of a line is expressed in terms of some unit of measure by means of an incommensurable number, that length is defined exactly. There cannot be two lengths expressed by the same incommensurable number. For if it were possible these lines must differ by some length, and any length, however small, will be greater than some fraction of the unit of measure. Let them differ by a length greater than $\frac{1}{1000}$ of the unit. Now let us approximate to the incommensurable number as far as thousandths, and let us suppose that it lies between 3·141 and 3·142. Then we shall have two lines, each greater than $3\frac{141}{1000}$ of the unit of measure, and each less than $3\frac{142}{1000}$ of the unit of measure differing by more than $\frac{1}{1000}$ of the unit; but this is clearly impossible. Hence only one length is expressed by an incommensurable number, when the unit is given.

We have extended the term 'number' from its original meaning of whole number first of all to fractions, and then to incommensurable numbers. We must now see if the same fundamental arithmetical laws will still hold true. Take for example the commutative law of addition, is the result the same whether we add $\sqrt{2}$ to $\sqrt{3}$, or $\sqrt{3}$ to $\sqrt{2}$?

We know that $\sqrt{2}$ is greater than $\frac{7}{5}$ and less than $\frac{3}{2}$, and that $\sqrt{3}$ is greater than $\frac{5}{3}$ and less than $\frac{7}{4}$. It follows that by adding $\sqrt{3}$ to $\sqrt{2}$ we obtain a sum greater than the sum of $\frac{7}{5}$ and $\frac{5}{3}$, and less than the sum of $\frac{3}{2}$ and $\frac{7}{4}$; that is to say, greater than $\frac{46}{15}$ and less than $\frac{26}{8}$. In the same way by taking continually closer approximations to the values of $\sqrt{2}$ and $\sqrt{3}$ we can find two fractions as nearly equal as we please between which the sum in question must lie. But we shall obtain exactly the same fractions if instead of adding $\sqrt{3}$ to $\sqrt{2}$ we add $\sqrt{2}$ to $\sqrt{3}$, since the commutative property is true for fractions. Now we have defined an incommensurable number when we have found fractions indefinitely near to one another between which that number must lie. Hence the sum of $\sqrt{2}$ and $\sqrt{3}$ must be equal to the sum of $\sqrt{3}$ and $\sqrt{2}$.

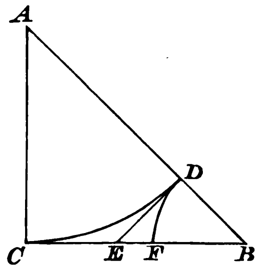
By the same method we could prove any other of the five fundamental laws. Incommensurable numbers therefore have the same general properties and may be treated by the same processes as whole numbers and fractions.

Note. The proof given above that the diagonal of a square is incommensurable with the side is found in the last proposition of the 10*th book of Euclid. It applies equally to the square roots of all numbers that are not exact squares. Another special proof is given in the same proposition, and is as follows. If possible let* $\sqrt{2}$ *be equal to a fraction in its lowest terms. Then the square of the numerator of that fraction will be double the square of the denominator, and will therefore be an even number. Hence the numerator itself must be an even number, for the square of an odd number is odd. Now the fraction is in its lowest terms. Therefore the denominator must be odd, or else the numerator and the denominator would not be prime to one another and the fraction would not be in its lowest terms. Again, since the numerator is divisible by* 2, *its square must be four times the square of some other number. But its square is only twice the square of the denominator. Therefore the square of the denominator is twice the square of this other number and is even. Hence the denominator itself is even. It is then both even and odd, which is absurd.*

This proof is of historical interest, since it probably represents the mode in which Pythagoras himself or his immediate followers proved the proposition.

The following proof, which requires a little knowledge of geometry, depends on the method of finding the greatest common measure.

ACB *is a right-angled triangle with the right angle at* C *and the sides* AC, CB *equal. By describing an arc of*



a circle we cut off AD *equal* AC. *Then any line which measures both* AC *and* AB *must measure* DB. *We draw* DE *at right angles to* DB, *and by drawing an arc of a circle cut* BF *equal to* BD. *Then* CE *is equal to* DE *or* DB *and* BF *is equal to* BD. *Hence by taking* 2BD *from* CB *we obtain* EF, *and any line which measures* AC *and* DB *must measure* EF. *But we can repeat exactly the same process with* EF *and* ED, *and however long we continue it there will still be a remainder. Hence no line measures both* AC *and* AB.

This proof gives all the approximation to $\sqrt{2}$ *already mentioned.* DB *is less than* $\frac{AC}{2}$ *therefore* AB *is less than* $\frac{3}{2}$AC. *For the same reason* EB *is less than* $\frac{3}{2}$DB, *therefore* CB *is less than* $\frac{5}{2}$DB *and* DB *is greater than* $\frac{2}{5}$CB. *Hence* AB *is greater than* $\frac{7}{5}$AC. *Again,* EB *is greater than* $\frac{7}{5}$ DB, CB *is greater than* $\frac{12}{5}$DB, DB *is less than* $\frac{5}{12}$CB *and* AB *is less than* $\frac{17}{12}$AC.

SECTION IV.

Ratio and Proportion continued.

THE definitions of ratio and proportion given in a former section are not applicable to incommensurable quantities. We must see how we can extend these definitions so as to include this new kind of quantities. We shall find that this is equivalent to defining the meaning of number in its widest sense. We will inquire then what signification we are to attach to the word "number" when we speak of 6, $\frac{2}{5}$ and $\sqrt{3}$ as "numbers".

In the first place the conception of number involves two quantities of the same kind; one quantity to be measured and one to serve as unit of measure. A line by itself or an interval of time by itself can have no relation to a number. They may be expressed by one, ten, a million, or any other number according to the unit chosen.

The number which expresses one quantity in terms of another quantity of the same kind is called the ratio of the first quantity to the second. When we have defined either of the terms "number", "ratio", we shall at the same time have defined the other. But "number", "ratio" are fundamental conceptions, and can only be defined by defining their essential relations. The first and most important of these relations is that of equality or identity. What do we mean when we speak of the number by which a length is expressed in terms of another length as identical with the number by which an interval of time is expressed in terms of another interval of time? In other words, what do we mean when we say the ratio of a first quantity to a second quantity is identical with the ratio of a third quantity to a fourth quantity? Such a relation between four quantities is called proportion. We see then that the explanation of the terms number and ratio will result from the definition of proportion. We may remark beforehand that in a proportion the first and second quantities must be of the same kind, and so also must the third and fourth, but it is not necessary for these two kinds to be identical. Remembering this we may give the following definition of proportion.

Four quantities are said to be in proportion if, when we divide the second into equal parts, and also the fourth into equal parts so that to every part of the second there corresponds a part of the fourth, and take several parts of the second and also several parts of the fourth, every part of the second again corresponding to a part of the fourth; then according as the whole quantity formed by the parts of the second is greater, equal, or less than t first, so is the whole quantity formed by the parts of t fourth greater, equal, or less than the third. The fi quantity is then said to be to the second as the third is the fourth.

If we assume as we legitimately may that the meani of equality of whole numbers is known (Introductic we can put the definition in a shorter form. The criteri of proportionality will then be as follows.

Divide the second quantity into equal parts and t fourth quantity into the same number of equal parts; ta any number of the parts of the second and the sai number of the parts of the fourth, then according as t whole formed by the parts of the second is greater, equ or less than the first, so the whole formed by the parts the fourth must be greater, equal, or less than the thi

We must next define what is meant by the sum two numbers.

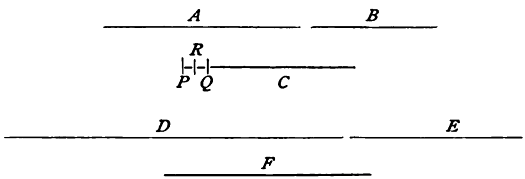
The sum of the numbers by which two quantities a expressed in terms of a third quantity is the number which the sum of the quantities is expressed in terms the third quantity.

But is the sum of two numbers independent of t unit of measure by means of which the quantities a expressed? If we take a fourth and fifth quantity (pressed in terms of the sixth by the same numbers as t first and second are in terms of the third, will the sum the fourth and fifth quantities be expressed in terms the sixth by the same number as the sum of the first a second is in terms of the third?

To prove this is the same as to prove the following theorem :

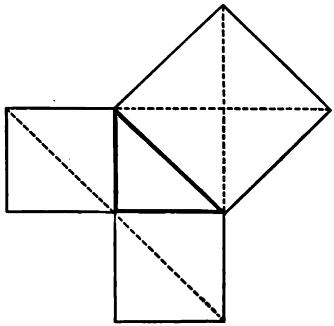
If we have six quantities, such that the first, third, fourth, sixth and also the second, third, fifth, sixth are proportional, then the sum of the first and second will be to the third as the sum of the fourth and fifth is to the sixth.

We assume what has been shewn in the introduction, that the sum of two whole numbers is independent of the special objects to which those numbers apply.



We represent the quantities by lines A, B, C, D, E, F. We divide C and F each into the same number of equal parts, say into 10 equal parts, and take, suppose, 13 of the parts of C and also 13 of the parts of F. And in the first place let the whole line formed by A and B be greater than the 13 parts of C, then we must shew that the whole line formed by D and E is greater than the 13 parts of F.

Let PQ be one of the parts of C. Then the line formed by A and B exceeds 13 times PQ by some quantity, and this quantity however small must be greater than some part of PQ. Let it be greater than the third



triangles and each of the small squares will contain two of these triangles. Therefore the large square is equal to the sum of the small squares and if the small squares be each a square inch, the large square will contain two square inches. Hence the square root of 2 which is commonly written $\sqrt{2}$ may be considered to be the length, expressed in inches of the longest side of a right-angled triangle, when each of the shorter sides is one inch; or in other words $\sqrt{2}$ may be considered to be the length of a diagonal of a square when each of the sides is 1.

Now it is obvious geometrically that according as a line is greater or less than another line, its square is greater or less than the square of that line; and conversely according as the square of a line is greater or less than the square of another line so is the first line greater or less than the second line. A line $\sqrt{2}$ inches in length must then be greater than any line whose square is less than 2 square inches, and less than any line whose square is

part of *PQ*. Divide *C* into 30 equal parts, each equal to *PR*, then *PQ* will contain 3 of these parts; and the line formed by *A* and *B* exceeds 13 times *PQ* or 39 times *PR* by a quantity greater than *PR*. Therefore this line is greater than 40 times *PR*. Now if we take successively 1, 2, 3, 4, &c. times *PR* we shall arrive at a multiple of *PR* just greater than *A*. Suppose then *A* is greater than 27 times *PR* and less than 28 times *PR*. Since *A* is less than 28 times *PR*, and the whole line is greater than 40 times *PR* it will follow that *B* is greater than 12 times *PR*. Now divide *F* into 30 equal parts, then since *A* is greater than 27 times *PR*, *D* will be greater than 27 of these parts, and since *B* is greater than 12 times *PR*, *E* will be greater than 12 of these parts. Hence the whole line formed by *D* and *E* is greater than 39 of the 30th parts of *F* and therefore greater than 13 of the 10th parts of *F*, since each 10th part contains three 30th parts.

Next let the line formed by *A* and *B* be less than 13 times the 10th part of *C*, then we must shew that the line formed by *D* and *E* is less than 13 times the 10th part of *F*.

Making the same suppositions as before, we shall have the line formed by *A* and *B* less than 38 times *PR*. Also if *A* is greater than 27 times *PR* and less than 28 times *PR*, *B* will be less than 11 times *PR*. Hence *D* is less than 28 times the 30th part of *F*, and *E* is less than 11 times the 30th part of *F*. Therefore the whole line formed by *D* and *E* is less than 39 times the 30th part of *F*, or than 13 times the 10th part of *F*.

Lastly, if the line formed by A and B is equal to 13 times the 10th part of C, the line formed by D and E is equal to 13 times the 10th of F For if that line were either greater or less than 13 times the 10th part of F, it would follow from what has gone before by interchanging A and B, D and E, C and F that the line formed by A and B was greater or less than 13 times the 10th part of C, and this is contrary to the supposition. The theorem is then completely proved.

We can extend the definition of addition so that it may apply to three or more numbers. If we take as many quantities as we please, then the sum of the numbers by which these quantities are expressed in terms of any unit of measure is the number by which the sum of these quantities is expressed in terms of the same unit of measure. It is easy to shew that this sum is independent of the particular unit of measure chosen. For if we first take two numbers and then add a third to their sum, and repeat the process till all the numbers are added, we shall by the preceding theorem at each step obtain a result independent of the unit of measure. It is also obvious that the sum is not affected by the order in which the numbers are added. The commutative and associative properties hold.

If the numbers added are all equal we shall obtain the multiple of a number. As a particular case of the preceding definition we may say that the multiple of the number by which any quantity is expressed in terms of a unit of measure is the number by which the same multiple of that quantity is expressed in terms of the same unit of measure. Also if we wish to divide a number into equal parts we must divide the corresponding quantity into equal parts, and find the number by which one of those parts is expressed. And lastly, to find any fraction of a number such as 13 times the 10th part of a number, we must find 13 times the 10th part of the corresponding quantity. We can prove without difficulty that in all these cases the number we obtain is independent of the unit of measure. For example, if the quantity A is expressed in terms of B by the same number as C is in terms of D, then the 4th part of A will be expressed in terms of B by the same number as the 4th part of C is in terms of D. For suppose the 4th part of A contains more than 3 times the 7th part of B, then A will contain more than 12 times the 7th part of B; hence C will contain more than 12 times the 7th part of D, and the 4th part of C will contain more than 3 times the 7th part of D. In the same way we shew that if the 4th part of A is equal to or less than 3 times the 7th part of B, the 4th part of C will be equal to or less than 3 times the 7th part of D.

When we are simply concerned with whole numbers, the multiplication of one number by another is a particular case of the addition of several numbers. But when we are treating of numbers generally this is not so, and we require an independent definition of multiplication.

To multiply one number by another assume any unit of measure and take a quantity expressed by the first number. Find another quantity which stands to this line in the ratio of the second number. Then the number by which this other line is expressed will be the product required.

That is to say, if A, B, C be three quantities, then the number which expresses B in terms of C multiplied by the number which expresses A in terms of B is equal to the number which expresses A in terms of C, or in other words the ratio of B to C multiplied by the ratio of A to B is equal to the ratio of A to C.

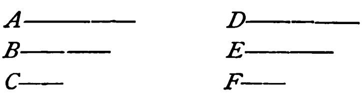
In order to establish the properties of multiplication we require the two following theorems.

1. If A is to B as C is to D then B is to A as D is to C.

2. The ratio of any fraction of A to the same fraction of B, for example, three-fifths of A to three-fifths of B, is equal to the ratio of A to B.

Their proof is left to the reader.

We must shew that, as in the case of addition, the number we obtain is independent of the particular unit of measure chosen. We must prove the following theorem. If A B, C, D, E, F be six quantities, and if the ratio of A to B be equal to the ratio of D to E and the ratio of B to C to the ratio of E to F, then the ratio of A to C will be equal to the ratio of D to F.



Represent these quantities by straight lines. Divide C and F into 10 equal parts, and suppose A is greater than 13 of the parts of *C*, then we must shew that *D* is greater than 13 of the parts of *F*. By however small a quantity *A* exceeds the 13 parts of *C* this quantity must be greater than some part of *B*. Let it be greater than the 9th part of *B*. Suppose the 13 parts of *C* to be greater than 5 of the 9th parts and less than 6 of the 9th parts of *B*. Then *A* will be greater than 6 of the 9th parts of *B*, and therefore *D* will be greater than 6 of the 9th parts of *E*. Now when *B* is to *C* as *E* is to *F* it is easy to shew that *C* is to *B* as *F* to *E*, and therefore the ratio of the 13 parts of *C* to *B* is the same as the ratio of the 13 parts of *F* to *E*. But the 13 parts of *C* are less than 6 of the 9th parts of *B*. Therefore the 13 parts of *F* are less than 6 of the 9th parts of *E*. And *D* is greater than 6 of the 9th parts of *E*. Hence *D* is greater than 13 of the 10th parts of *F*. The mode of completing the proof is obvious.

We must now see if multiplication so defined possesses the properties that hold for the multiplication of whole numbers. We begin with the commutative property. We must shew that the product of two numbers is unaltered, when the multiplier and multiplicand are interchanged.

Take four quantities in proportion *A*, *B*, *C*, *D*, and call the ratio of *A* to *B* the first number and the ratio of *B* to *C* the second number. Then the ratio of *A* to *C* is the result of multiplying the second number by the first. But the ratio of *C* to *D* is also the first number, and therefore the ratio of *B* to *D* is the result of multiplying the first number by the second. We have then to prove that A is to C as B is to D.

First, we will shew that when A, B, C, D are in proportion then according as B is greater, equal, or less than D so A is greater, equal, or less than C. Suppose B equal to D, A cannot be greater than C, for if possible let it exceed C by more than the 10th part of B or D. Let C contain more than 3 times and less than 4 times the 10th part of B. Then A will contain more than 4 times the 10th part of B. But it ought to contain less than 4 times the 10th part of B. Hence A cannot be greater than C and neither can C be greater than A. Therefore A and C are equal. Next suppose B greater than D. A cannot be equal to C, for then since B is to A as D is to C, B would be equal to D contrary to the supposition. Nor can A be less than C. For if possible let A be less than C by a quantity greater than the 10th part of D. Then if C is greater than 3 times and less than 4 times the 10th part of D, A will be less than 3 times the 10th part of D. But since B is greater than D, the 10th part of B is greater than the 10th part of D. Therefore A is less than 3 times the 10th part of B. But since C is greater than 3 times the 10th part of D, A ought to be greater than 3 times the 10th part of B. A cannot therefore be less than C, and since it was shewn that A cannot be equal to C it follows then A must be greater than C. Similarly if B were less than D, A would be less than C.

Now divide C and D into any number of parts, say 12, and take any number, say 5, of these parts. Then the following four quantities are in proportion, *A*, *B*, the 5 parts of *C* and the 5 parts of *D*. Hence according as *A* is greater than, equal to, or less than the 5 parts of *C*, so is *B* greater than, equal to, or less than the 5 parts of *D*. Therefore *A*, *C*, *B*, *D* are in proportion.

We must next consider the distributive law. We must shew that if we multiply a number by the sum of two numbers the result we obtain is the sum of the results we should have obtained if we had multiplied by each of the numbers separately. Let *C* represent in terms of some unit of measure the multiplied number, and let the ratios of *A* to *C* and *B* to *C* be the other two numbers. Then the ratio of the sum of the lines *A* and *B* to *C* will be the sum of these two numbers. Hence the whole line formed by joining *A* and *B* represents the product obtained when we multiplied *C* by this sum. But these lines *A* and *B* are separately the results obtained by multiplying *C* by each of the two numbers in turn. The whole product is therefore the sum of the separate products.

The associative property is as easily proved. Represent the three numbers by the ratios of *A* to *B*, of *B* to *C* and of *C* to *D*. The result of multiplying the third by the second is the ratio of *B* to *D*, and the result of multiplying this product by the first s the ratio of *A* to *D*. Again, the result of multiplying the second by the first is the ratio of *A* to *C*, and the result of multiplying the third by this product is the ratio of *A* to *D*. Whichever way we proceed the product of the three numbers will be the same.

We have now defined addition and multiplication and shewn that these operations possess for numbers generally the same fundamental properties as for whole numbers. We pass over subtraction since its definition and properties are obvious. We come then to division. To divide one number by another is to find a number which by multiplying the second number produces the first number. Hence the result of dividing the ratio of A to C by the ratio of B to C is the ratio of A to B.

Since numbers can be divided into equal parts and multiples of those parts can be taken, we can apply the definition of proportion to numbers as well as to concrete quantities. The ratio of two quantities will be equal to the ratio of the numbers by which these quantities are expressed in terms of any unit of measure. This can be shewn at once by using the definition of equality of ratio. But this ratio of two quantities is we have just seen the result of dividing the number which expresses one of the quantities by the number which expresses the other of the quantities. The result of dividing one number by another and the ratio of two numbers are therefore identical.

The numbers which express two quantities can be multiplied or divided by any number by altering the unit of measure. Hence the ratio of two numbers is unaltered if these numbers are both multiplied or divided by the same number.

The ratio of two numbers is often indicated by placing the sign : between the numbers. But it may also be expressed by writing one number over the other in the form of a fraction. Thus we can write the ratio of $\sqrt{6}$ to $\sqrt{15}$ either as $\sqrt{6} : \sqrt{15}$ or as $\frac{\sqrt{6}}{\sqrt{15}}$. Adopting the latter form we can call the first number the numerator and the second the denominator of the fraction. But it must be remembered that in this general case the fraction is a number of exactly the same kind as its numerator and denominator and only differs from them in the accidental form of its expression.

We can easily extend to these fractions the theorems proved true for ordinary fractions. Thus to multiply a fraction by any number we may multiply its numerator by that number. For example the product of $\frac{\sqrt{6}}{\sqrt{15}}$ and $\sqrt{7}$ is the same as the product of $\sqrt{6}$ and $\sqrt{7}$ divided by $\sqrt{15}$. To shew this we have only to shew that if we multiply the former product by $\sqrt{15}$, we obtain the product of $\sqrt{6}$ and $\sqrt{7}$. But this follows from the fact that the order in which the three numbers $\frac{\sqrt{6}}{\sqrt{15}}$, $\sqrt{7}$ and $\sqrt{15}$ are multiplied together is indifferent. Similarly we can shew that we divide a fraction by a number if we multiply its denominator by that number.

If four quantities in proportion be expressed by numbers in terms of any unit of measure we shall have four numbers in proportion. For example $\sqrt{6}$, $\sqrt{15}$, $\sqrt{14}$, $\sqrt{35}$ are in proportion. This relation is often written in the following way, $\sqrt{6} : \sqrt{15} :: \sqrt{14} : \sqrt{35}$. Or we may express both ratios as fractions and say $\frac{\sqrt{6}}{\sqrt{15}}$ is equal to $\frac{\sqrt{14}}{\sqrt{35}}$. All the properties proved for quantities in proportion will be true for numbers in proportion. Thus if

$$\sqrt{6} : \sqrt{15} :: \sqrt{14} : \sqrt{35},$$

we shall have $\sqrt{15} : \sqrt{6} :: \sqrt{35} : \sqrt{14}$,

and $\sqrt{6} : \sqrt{14} :: \sqrt{15} : \sqrt{35}$.

Of course when the fundamental properties of numbers are taken for granted, these properties of numbers in proportion can be proved easily. For example if $\frac{\sqrt{6}}{\sqrt{15}}$ is equal to $\frac{\sqrt{14}}{\sqrt{35}}$ we have, multiplying by the product of $\sqrt{15}$ and $\sqrt{35}$, the product of $\sqrt{6}$ and $\sqrt{35}$ equal to the product of $\sqrt{15}$ and $\sqrt{14}$; and dividing by the product of $\sqrt{14}$ and $\sqrt{35}$ we have $\frac{\sqrt{6}}{\sqrt{14}}$ equal to $\frac{\sqrt{15}}{\sqrt{35}}$. But it would be reasoning in a circle to attempt to prove the fundamental theorems relating to the proportion of quantities by expressing the quantities as numbers in terms of some unit of measure and then assuming the properties of numbers. For these latter properties depend on the very theorems it is desired to prove.

Euclid's definition of proportion is as follows :

Quantities are said to be in the same ratio, the first to the second and the third to the fourth when equal multiples of the first and third are at the same time greater equal or less than equal multiples of the second and fourth, whatever multiples be taken.

It is easy to see the connection of this definition with the one we have given. Suppose we have two squares and we wish to prove that the ratio of the diagonal to the side in one square is equal to the ratio of the diagonal to the side in the other square. We should divide the sides in 2, 3, 4, &c. parts and shew that the diagonals contained more than 2, 4, 5, &c., and less than 3, 5, 6, &c. of these parts. Euclid would shew that 2, 3, 4, &c. times the diagonal was in each case greater than 2, 4, 5, &c., and less than 3, 5, 6, &c. times the side. In arithmetical language we should define $\sqrt{2}$ by shewing that it was less than $\frac{3}{2}$ and greater than $\frac{7}{5}$, &c. Euclid would define $\sqrt{2}$ by shewing that twice $\sqrt{2}$ was less than 3, 5 times $\sqrt{2}$ greater than 7, &c. The two methods are at once seen to be equivalent.

SECTION VI.

Extraction of Square Roots.

In the extraction of square roots two cases present themselves. We may either wish to find the square root of a number which is an exact square or we may wish to approximate to the square root of a number which is not an exact square. We begin with the former case and we consider whole numbers first.

When the number is given us it is easy to tell how many figures there will be in the root. For the squares of all numbers less than 10 are less than 100, the square of all numbers less than 100 are less than 10,000, the square of all numbers less than 1,000 are less than 1,000,000.

Hence if the number contain 1 or 2 figures, its square root will contain 1 figure, if the number contain 3 or 4 figures, its square root will contain 2 figures, if the number contain 5 or 6 figures its square root will contain 3 figures.

When the root contains one figure we can only determine it from our knowledge of the squares of the first nine numbers. These squares must be remembered in extracting the square root, just as the multiplication table must be remembered in division.

When the number contains 3 or 4 figures we can also determine the root at once if the last two figures are 0. For instance the square root of 900 is 30, the square root of 6400 is 80.

Suppose now we have a number of 3 or 4 figures and with a square root therefore of 2 figures, and that the last two figures are not 0; for example the number 3969. We can find without difficulty the first figure of the root, for 3969 is greater than 3600 and less than 4900, and hence its root must be greater than 60 and less than 70. The first figure of the root must then be 6 or the root is 60 together with some one of the first nine numbers.

Now the square of 60 and any number is equal to the square of 60 together with twice the product of 60 and the number and the square of the number. The whole square is then the sum of 3600, 120 times the number and the square of the number.

But we wish the whole square to be 3969. Taking away 3600 we have then 120 times the number together with the square of the number equal to 369. Hence 369 is greater than 120 times the number and the number cannot be greater than 3. We try if 3 is the required number and we find that 120 times 3 is 360, and the square of 3 is 9. Therefore 3 is the figure in the unit's place and 63 is the square root of 3969.

Since 3 times 120 together with the square of 3 (or 3 times 3) is the same thing as 3 times 123, the work may be written thus:

```
      3969 (63
      3600
123)   369
       369
```

In finding the 6 we take no notice of the last two figures but see what is the greatest square contained in 39. This is 36 and therefore 3600 is the greatest square of a multiple of 10, which is less than 3969. We subtract 3600 from 3969 and find the remainder 369. We divide 369 by 120 the double of 60 and find the quotient 3. We add 3 to 120 and multiply the sum 123 by 3. This makes 369 and therefore 63 is the square root required.

The 0's can be left out as in division and we can write the work thus:

```
      3969 (63
      36
123)   369
       369
```

Two fresh figures are brought down at a time and not one as in division.

As another example we will find the square root of 205,209. Since this number has 6 figures the square root will have 3. We first of all find how many hundreds there are in the square root. We see that 20 is less than 25 the square of 5, and greater than 16 the square of 4. It follows that 205,209 must be less than 250,000 the square of 500 and greater than 160,000 the square of 400, and hence 4 must be the first figure of the square root. We may continue the work thus:

```
       205209 (453
       16
85)    452
       425
903)   2709
       2709
```

We subtract 16 the square of 4 from 20, the remainder is 4. We bring down two fresh figures, double the 4 to obtain 8 and then seek the quotient when 452 is divided by 80. This is 5, we add the 5 to the 80 and multiply 85 by 5. We subtract the product from 452 and bring down two fresh figures. We double the 5 and adding the result to the 80 obtain 90 which is the double of 45. We divide 2709 by 900 and obtain 3 as quotient. We add the 3 to the 900 and multiply 903 by 3. The result is 2709 and therefore 453 is the square root required.

To justify this process, we observe that the square root of 205209 must contain as many tens as the square root of 2052 does units. We can then proceed to extract the square root of 2052 in exactly the same way as in the former example where the number only contained four figures. We find that 2052 exceeds the square of 45 by 27.

Hence 205209 exceeds the square of 450 by 2709. Now the square of 450 and any other number is equal to the square of 450 together with twice the product of 450 and the other number and together with the square of the other number. It follows that 900 times the other number and the square of the other number must be together equal to 2709. Therefore the other number cannot be greater than 3. We try 3 and find that 3 times 900 and the square of 3 (or what is the same thing 3 times 903) are together equal to 2709.

Even in division we have to guess the successive figures of the quotient and we may not always find the right figure at the first trial. This is more frequently the case in extracting the square root since neither divisor nor quotient are exactly known. When it happens we must try the next lower figure which will generally be right.

As a last example we will find the square root of 80030916. The work will require no further explanation.

```
        80030916 (8946
        64
   169) 1603
        1521
  1784)   8209
          7136
 17886) 107316
        107316
```

We proceed to the approximate extraction of square roots and take as an example the square root of 2. The square root of 200 is ten times the square root of 2. Hence to find the square root of 2 to one place of decimals we need only divide the square root of 200 by 10, or what is the same thing, put a decimal point before the last figure of that square root. The work is then as follows:

2· (1·4
1·
2·4) 1·00
96
4

Hence 1·4 is approximately the square root of 2.

Again the square root of 2 is the hundredth part of the square root of 20,000; so that by bringing down two more o's we can find the square root to two decimal places.

2· (1·41
1
2·4) 1·00
96
2·81) 400
281
119

The reader who has worked out the examples on approximate multiplication and division will find no difficulty in extracting the square root of a number correctly to any required number of places. If we are working to 8 places the rule is simply to leave out all figures that would come in the succeeding decimal places. In consequence, after the first four figures have been found, the process comes to be identical with approximate division. The following example where the square root of 2 is found to 8 decimal places—7 of which ought to be correct—will be easily understood.

CHAPTER VII.

PROPERTIES OF NUMBERS. (*Continued.*)

SECTION I.

PERMUTATIONS. *Variations and Combinations.*

THE questions we shall consider in this section belong to an entirely new class and will be solved without employing any previous theorems.

The number of *permutations* of a set of objects is the number of ways in which all those objects can be arranged one after another.

Suppose the objects to be letters. The number of permutations of two letters *a* and *b* is two, since *ab* and *ba* are the only ways in which the letters can be written together.

If we have three letters *a*, *b*, *c* we can suppose each of them in turn to be the last letter. If *c* were the last letter we could write the letters *a* and *b* before it in the two ways already mentioned, namely *ab* and *ba*. In the same way if *b* were the last letter we could write the letters *a* and *c* before it in two ways *ac* and *ca*, and again, if *a* were the last letter we could write the letters *b* and *c* before it in two ways *bc* and *cb*. While still keeping any one of the letters *abc* last we have then two ways of writing these three letters, and therefore altogether the number of ways is 3 times 2 or 6. This will be seen clearly if we write the letters thus :

ab	*c*	*ac*	*b*	*bc*	*a*
ba	*c*	*ca*	*b*	*cb*	*a*

The vertical line is drawn separating the last letter from the two others so as to shew how the different permutations are formed.

We proceed in the same way for four letters *a*, *b*, *c*, *d*. We make *d* first of all the last letter and then we write the letters *a*, *b*, *c* before it in the six ways already given. But if *c* were the last letter we could put *a*, *b*, *d* before it in six ways and there would be six ways of writing *a*, *c*, *d* if *b* were the last letter and six ways of writing *b*, *c*, *d* if *a* were the last letter. Whichever of the letters *a*, *b*, *c*, *d* is last we have then six ways and therefore altogether 4 times 6 or 4 times 3 times 2 ways. The following mode of writing the four letters *a*, *b*, *c*, *d* shews clearly that the whole number of permutations is 4 times 3 times 2.

abc	*d*	*abd*	*c*	*adc*	*b*	*dbc*	*a*
bac	*d*	*bad*	*c*	*dac*	*b*	*bdc*	*a*
acb	*d*	*adb*	*c*	*acd*	*b*	*dcb*	*a*
cab	*d*	*dab*	*c*	*cad*	*b*	*cdb*	*a*
bca	*d*	*bda*	*c*	*dca*	*b*	*bcd*	*a*
cba	*d*	*dba*	*c*	*cda*	*b*	*cbd*	*a*

We have altogether four columns and each of these columns is divided by horizontal lines into three compartments containing two permutations. The whole number of permutations is then the product of the numbers 2, 3 and 4 or 24.

If we had five letters a, b, c, d, e by writing e as the last letter after the 24 permutations of a, b, c, d we should have 24 permutations of a, b, c, d, e. We should have likewise 24 permutations with d last, 24 with c last, 24 with b last, and 24 with a last. The whole number of permutations will therefore be 5 times 24 or 5 times 4 times 3 times 2.

The reader can easily continue this reasoning and shew that the number of permutations of 6 letters is the product of the numbers 2, 3, 4, 5, 6; the number of permutations of 7 letters is the product of the numbers 2, 3, 4, 5, 6, 7; and so on for any number of letters.

The ways in which we can take a given number of objects from a set of objects are called *variations* if we regard the order in which the objects are arranged.

For example we have ten messengers at our disposal, and we wish to send three of them to three different towns. The number of ways in which this can be done is called the number of variations of 10 things taken 3 together, provided we understand that if two messengers interchange towns it counts as a different way. Or again, we have five figures, say 1, 2, 3, 4, 5 written on five slips of paper, how many distinct numbers can we make by putting two of these slips together? This will be the number of variations of 5 things taken 2 together, since 12 and 21 are distinct numbers.

We will suppose our objects to be letters and we will write down two of the five letters *a*, *b*, *c*, *d*, *e* in as many ways as possible. Any one of the letters *a*, *b*, *c*, *d*, *e* may be the first letter. If *a* be the first we can write any one of the four letters *b*, *c*, *d*, *e* after it and shall thus have the four arrangements *ab*, *ac*, *ad*, *ae*. Similarly if *b* be the first we shall have the four arrangements *ba*, *bc*, *bd*, *be*; if *c* be the first the four arrangements *ca*, *cb*, *cd*, *ce*; if *d* be the first the four arrangements *da*, *db*, *dc*, *de*; and if *e* be the first the four arrangements *ea*, *eb*, *ec*, *ed*. We have then altogether 5 times 4 or 20 variations which may be written thus:

ab	*ba*	*ca*	*da*	*ea*
ac	*bc*	*cb*	*db*	*eb*
ad	*bd*	*cd*	*dc*	*ec*
ae	*be*	*ce*	*de*	*ed*

Next let us find the number of variations of 5 letters taken 3 together. Take any one of the arrangements of two letters; we can write after it any one of the three remaining letters to form an arrangement of three letters. For instance after *bd* we can write any one of the letters *a*, *c*, *e* and thus form the three variations *bda*, *bdc*, *bde*; and after *ec* we can write any one of the letters *a*, *b*, *d* and form the three variations *eca*, *ecb*, *ecd*. We have then for each of the 20 variations of 5 letters taken 2 together, 3 variations of 5 letters taken 3 together, and therefore there are altogether 20 times 3 or 5 times 4 times 3 variations of 5 letters taken 3 together. These 60 variations may be written thus;

abc	*bac*	*cab*	*dab*	*eab*
abd	*bad*	*cad*	*dac*	*eac*
abe	*bae*	*cae*	*dae*	*ead*
acb	*bca*	*cba*	*dba*	*eba*
acd	*bcd*	*cbd*	*dbc*	*ebc*
ace	*bce*	*cbe*	*dbe*	*ebd*
adb	*bda*	*cda*	*dca*	*eca*
adc	*bdc*	*cdb*	*dcb*	*ecb*
ade	*bde*	*cde*	*dce*	*ecd*
aeb	*bea*	*cea*	*dea*	*eda*
aec	*bec*	*ceb*	*deb*	*edb*
aed	*bed*	*ced*	*dec*	*edc*

In this way it is seen clearly that the whole number of variations is the product of the three numbers 5, 4, 3.

If we wish to find the whole number of variations of 5 letters 4 together we need only take any one of the variations of 5 letters 3 together such as *cae* and join on to it either of the two remaining letters *b* and *d*. We thus obtain two variations *caeb* and *caed*. The whole number of variations of 5 letters 4 together is therefore twice the number of variations of 5 letters 3 together. It is then the product of the four numbers 5, 4, 3, 2.

Lastly, if we intend to write down the whole number of variations of 5 things taken altogether, which is the same thing as the number of permutations of 5 letters, we may take any one of the variations of 5 letters 4 together and write after it the remaining letter. The number of variations will be unaltered and so we see in a different way that the number of permutations of 5 letters is 5 times 4 times 3 times 2.

We will now take ten letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *j*, and we will see how many variations we can make of these letters, first of all when two are taken at a time. We put any one of the ten letters, *a* for example, first and writing after it in turn the other 9 letters we obtain 9 variations, namely *ab*, *ac*, *ad*, *ae*, *af*, *ag*, *ah*, *ai*, *aj*. There are thus 9 variations whichever of the 10 letters is put first and therefore altogether 10 times 9 or 90 variations. These may be written thus;

ab	*ba*	*ca*	*da*	*ea*	*fa*	*ga*	*ha*	*ia*	*ja*
ac	*bc*	*cb*	*db*	*eb*	*fb*	*gb*	*hb*	*ib*	*jb*
ad	*bd*	*cd*	*dc*	*ec*	*fc*	*gc*	*hc*	*ic*	*jc*
ae	*be*	*ce*	*de*	*ed*	*fd*	*gd*	*hd*	*id*	*jd*
af	*bf*	*cf*	*df*	*ef*	*fe*	*ge*	*he*	*ie*	*je*
ag	*bg*	*cg*	*dg*	*eg*	*fg*	*gf*	*hf*	*if*	*jf*
ah	*bh*	*ch*	*dh*	*eh*	*fh*	*gh*	*hg*	*ig*	*jg*
ai	*bi*	*ci*	*di*	*ei*	*fi*	*gi*	*hi*	*ih*	*jh*
aj	*bj*	*cj*	*dj*	*ej*	*fj*	*gj*	*hj*	*ij*	*ji*

In order to write the variations of 10 letters taken three at a time, we should take any one of the above arrangements, *fe* for example, and write after it in turn the 8 remaining letters. We should thus obtain 8 variations, namely *fea*, *feb*, *fec*, *fed*, *feg*, *feh*, *fei*, *fej*. The whole number of variations of 10 letters taken 3 at a time will then be 8 times 90 or the product of the three numbers 10, 9, 8.

If we want the variations of 10 letters taken 4 at a time we take any one of the variations of 10 letters taken 3 at a time and join on to it in turn the 7 remaining letters. The number of variations is thus increased 7 times and is the product of the four numbers 10, 9, 8, 7.

The reader will now without further explanation be able to see the truth of the following general rule;

Take as many consecutive numbers as the number of letters to be grouped together, and let the highest of these numbers be the whole number of letters at our disposal. By multiplying these numbers together we shall have the required number of variations.

Thus to find the number of variations of 12 letters taken 5 at a time we must take the 5 consecutive numbers of which 12 is the highest, namely 12, 11, 10, 9, 8, and multiply them together.

The ways in which we can take a given number of objects from a set of objects are called *combinations* when the order in which the objects are arranged is indifferent.

For example, we have ten messengers at our disposal and we wish to send three of them together to the same place. The number of ways in which we can select these three is called the number of combinations of 10 things taken 3 together. Or again, the number of products we could form by multiplying two of the numbers 1, 2, 3, 4, 5 together would be the number of combinations of 5 things taken two at a time since the product of 3 and 4 and the product of 4 and 3 are the same.

We will once more suppose our objects to be letters, and we will find the number of combinations of the 5 letters *a*, *b*, *c*, *d*, *e* taken two at a time. Since *de* and *ed*, for example, count as the same combination but as two variations, and the same is true for every other combination, it is clear that the number of combinations must be half the number of variations. This latter number was however found to be the product of 5 and 4. The number of combinations must then be the product of 5 and 4 divided by 2, that is to say 10.

Again, let us find the number of combinations of 10 letters taken 3 at a time.

Let *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *j* be the letters. If we select any three of these letters, *dfh* for instance, we can form out of them several distinct variations, namely *dfh*, *dhf*, *fdh*, *fhd*, *hdf*, *hfd*, and the same will be true for any other three letters. The whole number of variations we can form from sets of 3 letters will be the number of permutations of 3 letters, that is to say, the product of 2 and 3. But the three letters *dhf* form only one combination. Therefore the number of *combinations* of 10 letters taken 3 at a time is obtained by dividing the number of *variations* of 10 letters, taken 3 at a time, by the product of 2 and 3. The latter number is however the product of 10, 9 and 8. The number of combinations is then the product of 10, 9, and 8 divided by the product of 2 and 3.

It is usual to say that the number of combinations is the product of the number 10, 9, 8 divided by the product of the numbers 1, 2, 3. Of course the product of 1, 2 and 3, and the product of 2 and 3 are the same thing, but by introducing the 1 we have three numbers to divide by, as well as three numbers to multiply together, and this makes the result easier to remember.

Now consider the number of combinations of 10 letters taken 4 at a time. For every one of these combinations such as *bdfh* we have a number of variations equal to the number of permutations of 4 letters; that is to say, to the product of 1, 2, 3 and 4. But the whole number of variations is the product of 10, 9, 7, 8. Hence the number of combinations must be the product of the numbers 10, 9, 7, 8 divided by the product of the numbers 1, 2, 3, 4.

The following then is the general rule for finding the number of combinations in any case;

Multiply together as many consecutive numbers as there are things to be taken at a time, and let the highest of these numbers be the whole number of things at our disposal. Divide the product by the product of as many numbers from 1 upwards as there are things to be taken at a time.

This result shews the truth of the following theorem which has already been otherwise proved;

If we multiply together as many consecutive numbers as we choose and also multiply together an equal number of the first numbers of the numerical series, then the first product will be divisible by the second.

For example, the product of the numbers 14, 15, 16, 17, 18, 19 is divisible by the product of the numbers 1, 2, 3, 4, 5, 6. In fact, the quotient must be the number of combinations of 19 things taken 6 at a time.

We have given a method of proving the rule for finding the number of variations in any case. But we can also deduce that result from the rule for finding the number of permutations of any number of letters.

We wish to find the variations of 10 letters taken 4 together. Imagine all the permutations of the 10 letters written down. Every arrangement of 4 letters will form in its turn the first four letters. But each arrangement will come first several times. Thus the arrangement *dghf* will occur first as many times as we can write after it the remaining letters *abceij*, that is to say, as many times as there are permutations of 6 letters. Hence the number of variations of 10 letters taken 4 together is equal to the number of permutations of 10 letters divided by the number of permutations of 6 letters. It is equal to the product of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by the product of the numbers 1, 2, 3, 4, 5, 6. This is the same thing as the product of the numbers 7, 8, 9, 10.

We have seen that to find the number of combinations of 10 things taken 4 together we must divide the number of permutations of 10 things taken 4 together by the product of the numbers 1, 2, 3, 4. Hence this number of combinations is equal to the product of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 divided by the product of the numbers 1, 2, 3, 4, 5, 6 and also by the product of the numbers 1, 2, 3, 4.

If we were to seek the number of combinations of 10 things taken 6 together we should obtain the same result.

In fact whenever we take 4 letters from 10 letters there must be 6 letters left, and therefore to every combination containing 4 letters there must correspond a combination containing 6 letters.

Another problem in permutations is to find the number of ways in which the letters of a word can be written when some of these letters occur more than once.

In how many ways for instance can the word *otto* be written?

We take a word of the same number of letters but with the letters all different, such as *into*, and write down its permutations.

into	*nito*	*tino*	*oint*
inot	*niot*	*tion*	*oitn*
iton	*ntio*	*tnio*	*onit*
itno	*ntoi*	*tnoi*	*onti*
iotn	*noit*	*toin*	*otin*
iont	*noti*	*toni*	*otni*

Now alter the *n* into *t* in all these permutations. We shall have every new permutation occurring twice over. Thus *into* and *itno* will both become *itto*; *inot* and *iton* will both become *itot*; *nito* and *tino* will both become *tito*, &c. The number of distinct permutations of *itto* must then be half the number of permutations of *into*; since for every permutation of *itto* there will be two permutations of *into* obtained by interchanging *n* and *t*. The permutations of *itto* are therefore,

itto, *itot*, *iott*; *tito*, *tiot*, *ttio*, *ttoi*, *toit*, *toti*;
oitt, *otit*, *otti*.

If we now make the *i* into *o* the number of permutations will again be halved, since to every permutation of *otto* will correspond two of *itto* obtained by interchanging the *o* and the *i*. Thus *itto* and *otti* will both become *otto*, *itot* and *otit* will both become *otot*. We have then the following six permutations of *otto*,

otto, *otot*, *oott*; *toto*, *toot*, *ttoo*.

The reader can extend this method to cases in which three or more letters become identical.

The rules for finding permutations and combinations were not known to the Greeks. They were discovered by the Hindoos and are given by a writer on astronomy, Bhascara, who lived in the 12th century.

SECTION II.

ARITHMETICAL AND GEOMETRICAL PROGRESSION.

WHEN the difference between any two successive numbers in a series is constant, the series is said to be an arithmetical progression. Thus numbers such as 2, 5, 8, 11, 14, 17 are said to be in arithmetical progression, since each number is greater by 3 than the number immediately preceding it. The different numbers in a series are called its *terms*. In the example given 2 is the first term, 5 the second term, 8 the third term, &c. The interval between any two consecutive terms of the series is called the *common difference*. The common difference in the present case is 3.

When we know the first term and the common difference of an arithmetical progression we can at once find any other term. Thus, if 2 be the first term, 3 the common difference, the 2nd term will be 5 or 2 and 3, the 3rd term 8 or 2 and twice 3, the 4th term 11 or 2 and 3 times 3, the 5th term 14 or 2 and 4 times 3, the 6th term 17 or 2 and 5 times 3. Hence the 10th term will be 2 and 9 times 3 or 29, the 13th term 2 and 12 times 3 or 38.

Or again, take the series 6, 8, 10, 12, &c. and let us find the 20th term. It will be 6 and 19 times 2 or 44.

There is a simple method of finding the sum of any number of terms of an arithmetical progression. To find for example the sum of the numbers 2, 5, 8, 11, 14, 17, we write these numbers twice over and arrange them in two rows, in the first row in their natural order, and in the second row backwards; thus

$$\begin{array}{l} 2,\ \ 5,\ \ 8,\ 11,\ 14,\ 17 \\ \underline{17,\ 14,\ 11,\ \ 8,\ \ 5,\ \ 2} \\ 19,\ 19,\ 19,\ 19,\ 19,\ 19 \end{array}$$

Add each number to the one above it and we always obtain 19. The sum of the two rows must then be equal to 6 times 19. But since apart from their order the numbers in the two rows are the same, this sum must be equal to twice the sum of the numbers in either row. The required sum is then half the product of 6 and 19 or 57.

Again, let us find the sum of 20 terms of the series 6, 8, 10, 12, &c. The last term we saw was 44, the term before will be two less or 42, &c. Write the series both ways and add;

$$\begin{array}{l} 6,\ \ 8,\ \ \ldots\ldots\ldots\ldots\ \ 42,\ 44 \\ \underline{44,\ 42,\ \ \ldots\ldots\ldots\ldots\ \ \ 8,\ \ 6} \\ 50,\ 50,\ \ \ldots\ldots\ldots\ldots\ \ 50,\ 50 \end{array}$$

Since the numbers in the first row constantly increase by 2 and the numbers in the second row constantly diminish by 2, the sum of two corresponding numbers will be always the same number, namely 50. It follows that twice the sum we require is 20 times 50 and the sum itself is 500.

We have then the following rule;

Add the first and the last terms and multiply by the number of terms. Half the product will be the sum of the series.

An arithmetical progression must clearly consist of at least three terms. When it consists of no more, the middle term is said to be the arithmetical mean of the first and last terms. Thus in the progression 2, 5, 8, 5 is said to be the arithmetical mean of 2 and 8. The arithmetical mean of two numbers is always half their sum. For example, the arithmetical mean of 5 and 21 is the half of 26 or 13.

When in a series of numbers each number bears a constant ratio to the preceding number, the series is said to be a geometrical progression. Thus the numbers 2, 6, 18, 54, 162 are said to be in geometrical progression, since 6 is 3 times 2, 18 is 3 times 6, 54 is 3 times 18, &c. The number 3 is called the common ratio.

To find the sum of these numbers 2, 6, 18, 54, 162, we multiply each number and therefore the whole sum by 3 the common ratio. We shall have 3 times the required sum equal to the sum of the numbers 6, 18, 54, 162, 486. Subtract from this sum the original numbers. We see that 6, 18, 54, 162 are common to both sets and therefore twice the required sum is equal to the difference of 486 and 2. The sum is therefore 242.

Again, take the numbers 1, 5, 25, 125, 625.

Five times this sum is the sum of 5, 25, 125, 625, 3125.

Hence four times their sum is the difference 3125 and 1 and the sum is 781.

The common ratio need not be always a whole number. Thus 4, 6, 9 are in geometrical progression and the common ratio is $\frac{3}{2}$.

When three numbers are in geometrical progression the middle number is said to be the geometrical mean of the first and last number. Thus 6 is the geometrical mean of 4 and 9. When we wish to find the geometrical mean of two numbers we multiply them together and extract the square root of their product. If 5 and 45 be the numbers, their product will be 225 and the square root 15, and we can see that 5, 15, 45 are in geometrical progression.

It is possible to have a geometrical progression in which the terms continually decrease. Such for example is the series of numbers $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$.

Three times the sum of this series will be the sum of the number 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, and therefore twice the sum will be the difference of 1 and $\frac{1}{81}$. We should find in the same way that twice the sum of $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$ was equal to the difference of 1 and $\frac{1}{243}$, and that twice the sum of six numbers was equal to the difference of 1 and $\frac{1}{1029}$, and so on; the more numbers we take the nearer twice the sum becomes to 1 and by taking enough numbers we can make it become as near 1 as ever we please. This is expressed shortly by saying that twice the sum of the infinite series of $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. is equal to 1 or that the sum itself is equal to $\frac{1}{2}$.

Recurring decimal fractions are a case of an infinite series of numbers in geometrical progression.

Arithmetical and geometrical progressions were known to the Pythagoreans. Euclid shews in the 35th proposition of his 9th book how to sum a series of numbers in geometrical progression.

SECTION III.

Figurate Numbers.

The following numbers are called *figurate*:

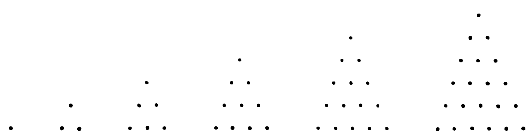
1, 2, 3, 4, 5, 6, 7, 8, 9, 10...
1, 3, 6, 10, 15, 21, 28, 36, 45, 55...
1, 4, 10, 20, 35, 56, 84, 120, 165, 220...
1, 5, 15, 35, 70, 126, 210, 330, 495, 715...
1, 6, 21, 56, &c. ....
....

They are formed thus:

We first write the series of numbers in their natural order for as many terms as we please. We begin the second series with 1 and form each term successively by adding to the preceding term the number that follows the number above it in the first series. Thus 1 and 2 give 3, 3 and 3 give 6, 6 and 4 give 10, &c. In the same way each successive series is derived from the preceding. For the third series we should say 1 and 3 give 4, 4 and 6 give 10, 10 and 10 give 20, &c. The numbers in the second series are called *triangular*, those in the third series are called *pyramidal*. In the remaining series the numbers have no special denomination but only the general name *figurate*.

From the mode of formation it is clear that any number in any series is equal to the sum of all the numbers up to and including the number immediately above it in the preceding series. Thus 21 is the sum of the numbers 1, 2, 3, 4, 5, 6, since we have formed the series 1, 3, 6, 10, 15, 21 by adding these numbers one after another, and 56 is for the same reason the sum of the numbers 1, 3, 6, 10, 15, 21.

The natural symbol for any number is not an arbitrary sign such as 3 or 4, but the required number of some kind of objects such as strokes or dots*. If we represent the triangular numbers by dots, we can arrange these dots in the following ways:



The first figure consists of one dot, the second figure of one and two dots, the third is formed from the second by adding three dots, the fourth from the third by adding four dots, &c. We thus always obtain a triangular number of dots, and the figures shew the reason of the name.

We cannot represent the pyramidal numbers by figures in a plane, but if the reader will take some

* This remark is made in the Arithmetic of Nicomachus of Gerasa already mentioned.

kind of small balls and put the first triangular number of balls on the second and then both together on the third, and all three on the fourth and so on, he will be able to arrange the balls in pyramids, and the number of balls will always be a pyramidal number.

We will now shew how to find any figurate number immediately without going through the process of successive addition.

We begin with the triangular numbers. The 7th triangular number is by its definition the sum of the first seven numbers and we have seen in the last section how to find such a sum. We write the numbers forwards and backwards and add, thus :

1,	2,	3,	4,	5,	6,	7
7,	6,	5,	4,	3,	2,	1
8,	8,	8,	8,	8,	8,	8

Twice the sum is then 7 times 8, and therefore the 7th triangular number is half the product of 7 and 8. Similarly the 9th triangular number is half the product of 9 and 10, the 4th is half the product of 4 and 5, &c.

There is another way in which we can prove this result. We take 8 letters and write out all their combinations two together. We first write those in which *a* enters, then those in which *b* enters, &c.

ab,	*ac*,	*ad*,	*ae*,	*af*,	*ag*,	*ah*
	bc,	*bd*,	*be*,	*bf*,	*bg*,	*bh*
		cd,	*ce*,	*cf*,	*cg*,	*ch*
			de,	*df*,	*dg*,	*dh*
				ef,	*eg*,	*eh*
					fg,	*fh*
						gh

a will enter into 7 combinations since it may be followed by any of the letters *b*, *c*, *d*, *e*, *f*, *g*, *h*. *b* will enter into 6 new combinations since it may be followed by any of the letters which come after it but not by *a*, for *ba* is the same combination as *ab*. *c* will enter into 5 new combinations, *d* into 4, *e* into 3, *f* into 2, *g* into 1. The total number of combinations is then the sum of 7, 6, 5, 4, 3, 2, 1 and this is the same thing as the 7th triangular number. But we know that the number of combinations of 8 things taken two together is the product of 8 and 7 divided by the product of 1 and 2. Hence the 7th triangular number is the product of 7 and 8 divided by the product of 1 and 2.

We can apply the same method to finding other figurate numbers. We will find the 7th pyramidal number or what is the same thing the sum of the first 7 triangular numbers.

We take 9 letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i* and write out their combinations 3 together. We first of all write those in which *a* comes first. If *a* be written before any of the combinations of the letters *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i* taken two together we shall have a combination of the letters *a*, *b*, *c*, &c. taken three together. Hence the total number of combinations in which *a* occurs will be the number of combinations of 8 things 2 together or the 7th triangular number. We next find the number of *new* combinations in which *b* occurs. This will be the number of combinations of the 7 letters *c*, *d*, *e*, *f*, *g*, *h*, *i* taken 2 together, or the 6th triangular number, since *a* cannot appear again. In the same way if we take *c d e f g* one after another, we shall see that the number of new combinations in which they will occur will be respectively the 5th, 4th, 3rd, 2nd, 1st triangular numbers. Hence the total number of combinations is the sum of the first seven triangular numbers. But the number of combinations of 9 things taken 3 together is the product of the numbers 9, 8, 7 divided by the product of the numbers 1, 2, 3. Therefore the sum of the first 7 triangular numbers, or the 7th pyramidal number, is equal to the product of 7, 8, 9 divided by the product of 1, 2, 3.

We may prove in the same way that the 10th pyramidal number is the product of 10, 11, 12 divided by the product of 1, 2, 3; that the 4th pyramidal number is the product of 4, 5, 6 divided by the product of 1, 2, 3, &c.

If we wish to find the sum of the first 7 pyramidal numbers or the 7th number in the 4th series we write out the combinations of 10 things taken 4 together. The number of combinations in which *a* occurs will be the number of combinations of 9 things taken 3 together or the 7th pyramidal number, the number of new combinations in which *b* occurs will be the 6th pyramidal number, &c. The total number of combinations will then be the sum of the first 7 pyramidal numbers. But this number of combinations is the product of 10, 9, 8, 7 divided by the product of 1, 2, 3, 4. Hence the 7th number of the 4th series is the product of the numbers 7, 8, 9, 10 divided by the product of the numbers 1, 2, 3, 4.

We can apply the same method to finding the number in other series.

We have seen that triangular numbers are formed by adding together the successive terms of the series 1, 2, 3, 4, 5, 6, 7, &c. where each term differs from the preceding by 1, and also that square numbers are formed by adding together the successive terms of the series 1, 3, 5, 7, 9, 11, &c. where each term differs from the preceding by 2. In the same way if we add the terms of the series,

1, 4, 7, 10, 13, 16, &c.

where each term differs from the preceding by 3, we obtain the following numbers, which are called *pentagonal*:

1, 5, 12, 22, 35, 41, &c.

By adding the terms of the series 1, 5, 9, 13, 17, 21, &c. we obtain the *hexagonal* numbers 1, 6, 15, 28, 45, 66, &c. and we can obtain other series of numbers in the same way. To all these numbers the name *polygonal* is given.

Triangular numbers were probably discovered by Pythagoras. Figurate numbers generally are treated of by several Greek writers, the chief of whom are Hypsicles, who lived in the 2nd century B.C., and Nicomachus of Gerasa. Diophantus, who lived in the end of the 3rd century of our era, wrote a special treatise on the subject of polygonal numbers.

SQUARES AND CUBES OF NUMBERS.

WE shall in this section give some fresh properties relating to squares and cubes, and shew how to find the sum of any number of squares or cubes of the natural numbers taken in order.

If in the series of triangular numbers

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78,

we add any triangular number to the triangular number immediately preceding it we shall have the corresponding square number. Thus the 5th triangular number 15 and the 4th triangular number 10 give 25 the square of 5, the 9th triangular number 45, and the 8th triangular number 36 gives 81 the square of 9, &c.

It is easily proved that this must always be the case. We saw for example that the 7th triangular number was half the product of 7 and 8, and the 6th triangular number was half the product of 6 and 7. Now half 8 times 7 together with half 6 times 7 must be half 14 times 7 or 7 times 7. Hence the two triangular numbers together make the square of 7.

Another way of proving the theorem is to write the triangular numbers each as a sum of natural numbers, but the second series of numbers must be written backwards. We then add each number to the number immediately below, thus :

the 7th triangular number is the sum of	1, 2, 3, 4, 5, 6, 7
„ 6th „ „ „	6, 5, 4, 3, 2, 1
the square of 7 is the sum of	7, 7, 7, 7, 7, 7, 7

Or we may say that by running up and down the series of numbers we obtain a square number. The numbers 1, 2, 3, 4, 5, 4, 3, 2, 1 are equal to the square of 5.

Now let us find the sum of the first seven square numbers. We have the following relations:

The square of

7 = the sum of the 7th and 6th triangular numbers.						
6	...	... 6	...	5	...	...
5	...	... 5	...	4	...	...
4	...	... 4	...	3	...	...
3	...	... 3	...	2	...	...
2	...	... 2	...	1	...	...
1 = the 1st triangular number.						

Adding these equalities we have the theorem that the sum of the first seven square numbers is equal to the sum of the 7th and 6th *pyramidal* numbers.

Now the 7th pyramidal number is equal to one-sixth of the product of 7, 8, 9 and the 6th pyramidal number is equal to one-sixth of the product of 6, 7, 8. But 6 times the product of 7 and 8, and 9 times the product of 7 and 8 together make 15 times the product of 7 and 8. Hence the sum of the first 7 square numbers is equal to the product of 7, 8 and 15 divided by 6.

Let us now find the sum of the first 10 square numbers. We prove as before that it is equal to the sum of the 10th and 9th pyramidal numbers. But the 10th pyramidal number is equal to one-sixth of the product of 10, 11, 12 and the 9th pyramidal number is equal to one-sixth of the product of 9, 10, 11. Now 12 times the product of 10 and 11 together with 9 times the product of 10 and 11 makes

21 times the product of 10 and 11. Hence the sum of the first 10 square numbers is the product of 10, 11 and 21 divided by 6.

The following then is the rule:

Take the last number, the next following number and their sum. Multiply these numbers together and divide by 6. The result will be the required sum of the squares.

Thus to find the sum of the first 13 square numbers we take 13, 14 and their sum 27. The 6th part of the product of 13, 14 and 27 will be the sum required.

If we construct a multiplication table up to any point, say 8 times 8, and divide it into compartments as in the figure below, we shall see that each compartment will contain the cube of a number.

1	2	3	4	5	6	7	8
2	4	6	8	10	12	14	16
3	6	9	12	15	18	21	24
4	8	12	16	20	24	28	32
5	10	15	20	25	30	35	40
6	12	18	24	30	36	42	48
7	14	21	28	35	42	49	56
8	16	24	32	40	48	56	64

Thus the first compartment contains 1 the cube of 1, the second compartment contains 2, 4 and 2 which together make 8 the cube of 2, the third compartment contains 3, 6, 9, 6, 3 which together make 27 the cube of 3, &c.

This theorem results immediately from the theorem already proved, that the numbers written in order from 1 up to any number and then backwards down to 1 again together make the square of the highest number. The numbers 1, 2, 3, 4, 5, 4, 3, 2, 1 together make the square of 5. Now multiply each number by 5 and then we see that the cube of 5 is equal to the sum of the numbers 5, 10, 15, 20, 25, 20, 15, 10, 5.

Or we may write the number in any compartment, the 7th for instance in the following way;

7,	14,	21,	28,	35,	42,	49
42,	35,	28,	21,	14,	7	
49,	49,	49,	49,	49,	49,	49

By adding them we have 7 times 49, that is to say the cube of 7.

It follows that the sum of the first 8 cubes is equal to the sum of all the numbers in a multiplication table constructed up to 8 times 8.

To put this result in a simpler form we must employ the following theorem:

If we multiply one sum of several numbers by another sum of several numbers, the total product will be the sum of the partial products we should obtain if we multiplied each number in the first sum by each number in the second sum.

Thus if we multiply 13 the sum of 4, 6 and 3, by 11 the sum of 4, 2 and 5, the product will be the sum of the following nine partial products:

4 times 4, 4 times 6, 4 times 3;
2 times 4, 2 times 6, 2 times 3;
5 times 4, 5 times 6, 5 times 3.

For to give first 4 objects, then 6 objects and then 3 objects to each of 4 people is the same as giving each of the 4 people, 13 objects at once. Hence the first three products together make 4 times 13. In the same way the next three products are equal to 2 times 13, and the last three products to 5 times 13.

But 4 and 2 and 5 thirteens are equal to 11 thirteens just as 4 and 2 and 5 objects of any kind are equal to 11 objects. Therefore the sum of the products is equal to 11 times 13.

Now the multiplication table up to 8 times 8 contains all the partial products obtained by multiplying the numbers 1, 2, 3, 4, 5, 6, 7, 8 by the same numbers 1, 2, 3, 4, 5, 6, 7, 8. Hence the sum of the numbers in this multiplication table must be equal to the product obtained when we multiply the sum of the numbers 1, 2, 3, 4, 5, 6, 7, 8 by itself, that is to say it must be equal to the square of this latter sum.

We see then that the sum of the first 8 cubes is equal to the square of the sum of the first 8 numbers, and the theorem will be true for any other number of cubes.

We can apply this result to prove the theorem of Nicomachus given in a former chapter. We know that the sum of the first two odd numbers is equal to the square of 2; the sum of the first three odd numbers to the square of 3, &c. Hence if we write down first 1, then 2, then 3, then 4, then 5, then 6 odd numbers, thus:

1 | 3, 5 | 7, 9, 11 | 13, 15, 17, 19 | 21, 23, 25, 27, 29 | 31, 33, 35, 37, 39, 41 |

the sum of all these numbers will be equal to the square of the sum of the numbers 1, 2, 3, 4, 5, 6. It will therefore be equal to the sum of the cubes of the numbers 1, 2, 3, 4, 5, 6. In the same way the sum of the first five sets of odd numbers is equal to the sum of the cubes of the numbers 1, 2, 3, 4, 5. Hence the last set containing 6 odd numbers must be equal to the cube of 6.

The reader who has some acquaintance with Algebra may prove the theorems in the last two chapters by the same methods, but with the use of algebraical notation. He will thus be able to appreciate the advantages of that notation in respect of ease and rapidity.

EXAMPLES.

1. How many different straight lines can join five points when no three points lie on the same straight line?

2. If these lines be produced when necessary so that they all intersect, how many fresh points of intersection will be formed?

3. How many triangles can be formed from five points, and how many from eight points?

4. How many tetrahedra can be formed from eight points, no four of which are in the same plane?

5. In how many different ways can the letters in the words *essay* and *assay* be arranged?

6. Prove Euclid's rule for the formation of perfect numbers.

7. What are the 10th hexagonal and 8th heptagonal numbers?

8. Prove that the sum of the 10th hexagonal and 9th triangular numbers is equal to the 10th heptagonal number.

9. If binary fractions were used instead of decimal fractions, how would one-third and one-fifth be expressed?

ANSWERS.

Answers to Examples on Chapter I.

1.

1	2	3	4	5	6	7	8	9	10	11	12
a	*b*	*c*	*bb*	*d*	*bc*	*e*	*bbb*	*cc*	*bd*	*f*	*bbc*

13	14	15	16	17	18	19	20
g	*be*	*cd*	*bbbb*	*h*	*bcc*	*i*	*bbd*

2. Half a dozen, Half a million, Duodeviginti (2 from 20).

3. Four hundred and nineteen.

4. Six gross and three units.

5. The number of words we shall want for

2	3	4	5	6	7	8	9	10	11	12	will be
11	8	8	8	9	9	10	11	12	12	13	

6.

I,	II,	III,	IIII,	V,	VI,	VII,	VIII,	VIIII,	VV,	VVI,	VVII,
1	2	3	4	5	6	7	8	9	10	11	12

VVIII,	VVIIII,	VVV,	VVVI,	VVVII,	VVVIII,	VVVIIII,	VVVV,
13	14	15	16	17	18	19	20

VVVVI,	VVVVII,	VVVVIII,	VVVVIIII.
21	22	23	24

7. Using the letters of our Alphabet

a	*b*	*c*	*d*	*e*	*ea*	*eb*	*ec*	*ed*	*f*	*fa*	*fb*	*fc*	*fd*	*g*
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

ga	*gb*	*gc*	*gd*	*h*	*ha*	*hb*	*hc*	*hd*
16	17	18	19	20	21	22	23	24.

8. 11110011. 9. 117. 10. 2454. 11. 305.

12. 3017. 13. 573 and 900 respectively.

ANSWERS TO EXAMPLES ON CHAPTER II.

3.
```
 4827147
 5186991
 2293656
 2143812
--------
13239496
 121211
--------
14441506
   1 1
--------
14451606
```

4.
```
7926321
6697775
-------
1339656
  11111
-------
1228546
```

5.
```
229365000
229365000
---------
458730000
```

```
 2293650
 2293650
 2293650
 2293650
 2293650
 2293650
--------
13761900
```

```
229365
229365
229365
------
688095
```

```
458730000
 13761900
   688095
---------
473179995
```

6.

389748		
263000 −	1000	
126748		
26300 −	100	
100448		
26300 −	100	
74148		
26300 −	100	
47848		
26300 −	100	
21548		
2630 −	10	
18918		
2630 −	10	Quotient 1481
16288		Remainder 245
2630 −	10	
13658		
2630 −	10	
11028		
2630 −	10	
8398		
2630 −	10	
5768		
2630 −	10	
3138		
2630 −	10	
508		
263 −	1	
245		

7. 1100001001.

8. 1000010.

9. 111101010.

10. In 7, the numbers to be added are 245, 179, 214, 139, and the answer is 777.

In 8, the numbers are 179 and 245, and the answer is 66.

In 9, the numbers to be multiplied are 35 and 14, and the answer is 490.

ANSWERS TO EXAMPLES ON CHAPTER III.

2. 39; 29. 3. 2, 2, 2, 2, 5, 103. 4. 1.

7. No; it is the product of 23 and 37.

9. 75348. 10. 840. 12. 8128.

ANSWERS TO EXAMPLES ON CHAPTER IV.

1. $10\frac{17}{18}$. 2. $\frac{13}{36}$. 4. $1\frac{2}{29}$.

5. $3\frac{1}{7}$ is the greater. The difference is $\frac{1}{791}$.

6. $\frac{2}{17}$. 7. $\frac{2}{43}$. 8. $\frac{2}{79}$. 9. $\frac{1}{8}$.

Examples 5—9 are to be found in an Egyptian treatise on Arithmetic and Geometry written between 2000 B.C. and 1700 B.C.

ANSWERS TO EXAMPLES ON CHAPTER V.

1. ·5784057. 2. ·41706.

3. 4·4872091. 4. 10·3321808.

5. ·06820020. 6. ·453392312532.

7. 1·128348.

8. 2·3742.

9. ·18791808.

10. 50.

11. 18 and ·018.

12. 3652 square yards.

13. There may be in the 6th place of decimals an error of (*a*) 3 in excess, (*b*) 2 in defect or 1 in excess, (*c*) 1 in defect or 2 in excess, (*d*) 3 in defect.

ANSWERS TO EXAMPLES ON CHAPTER VI.

1. The square is ·02459, the cube is ·00386, the 4th power is ·00061, the 5th power is ·00009, the 6th power is ·00001. Higher powers will have no figure in the 5th decimal place.

2. Square ·1840166, cube ·0789379, 4th power ·0338622, 5th power ·0145207.

4. 1·1892071.

5. 1·73205.

6. 1·5; 1·4; 1·416; 1·414 nearly. Comparing with 1·4142... the errors are seen to occur in the 1st, 2nd, 3rd and 4th decimal places respectively.

7. The result is ·072535.

8. The result is ·031501 correct to the last decimal.

9. Square ·12588, cube ·04466, 4th power ·01584, 5th power ·00561, 6th power ·00198, 7th power ·00069, 8th power ·00024, 9th power ·00008.

10. ·13187 correct to the last decimal.

Answers to Examples on Chapter VII.

1. 10. 2. 15. 3. 10 and 56. 4. 70.

5. 60 and 30. 7. 190 and 148.

8. This is easily shewn by writing the numbers thus,

$$\begin{array}{rrrrcr} 1, & 5, & 9, & 13, & \ldots\ldots & 37 \\ & 1, & 2, & 3, & & 9 \\ \hline 1, & 6, & 11, & 16, & \ldots\ldots & 26 \end{array}$$

9. By ·010101... and ·0011001100 11..........

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